Foundations of Geometry

Chapter 1. The Axiomatic Method

1.7. Finite Geometries—Proofs of Theorems

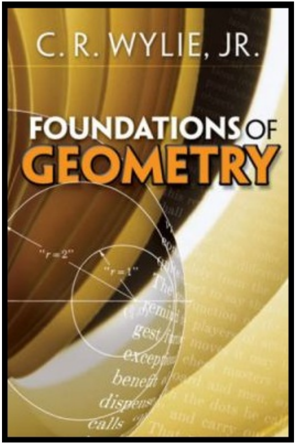


Table of contents

Theorem 1.7.1

Theorem 1.7.1. There exists at least one point.

Proof. By A.6 there exists at least one line ℓ. By A.4, line ℓ contains at least three points. Hence there is at least one point, as claimed. □

Theorem 1.7.1

Theorem 1.7.1. There exists at least one point.

Proof. By A.6 there exists at least one line ℓ. By A.4, line ℓ contains at least three points. Hence there is at least one point, as claimed. □

Theorem 1.7.2

Theorem 1.7.2. If ℓ_1 and ℓ_2 are any two lines, there is at most one point which lies on both ℓ_1 and ℓ_2.

Proof. Let ℓ_1 and ℓ_2 be any two lines. ASSUME that the two points P_1 and P_2 are on both ℓ_1 and ℓ_2. By A.2 there is at most one line containing any given two points, so this is a CONTRADICTION. So the assumption of two point shared by ℓ_1 and ℓ_2 is false, and hence ℓ_1 and ℓ_2 can share at most one point, as claimed. □

Theorem 1.7.2

Theorem 1.7.2. If ℓ_1 and ℓ_2 are any two lines, there is at most one point which lies on both ℓ_1 and ℓ_2.

Proof. Let ℓ_1 and ℓ_2 be any two lines. ASSUME that the two points P_1 and P_2 are on both ℓ_1 and ℓ_2. By A.2 there is at most one line containing any given two points, so this is a CONTRADICTION. So the assumption of two point shared by ℓ_1 and ℓ_2 is false, and hence ℓ_1 and ℓ_2 can share at most one point, as claimed. □

Theorem 1.7.3

Theorem 1.7.3. Two points determine exactly one line.

Proof. Let P_1 and P_2 be two points. By A.1, there is a line ℓ containing both P_1 and P_2. By A.2, there is not another line containing both P_1 and P_2, so ℓ is the exactly one line containing these two points. □

Theorem 1.7.3

Theorem 1.7.3. Two points determine exactly one line.

Proof. Let P_1 and P_2 be two points. By A.1, there is a line ℓ containing both P_1 and P_2. By A.2, there is not another line containing both P_1 and P_2, so ℓ is the exactly one line containing these two points. □

Theorem 1.7.4

Theorem 1.7.4. Two lines have exactly one point in common.

Proof. Let ℓ_1 and ℓ_2 be lines. By A.3, there is a point P which lies on both ℓ_1 and ℓ_2. By Theorem 1.7.2, there is at most one point which lies on both ℓ_1 and ℓ_2. Therefore, there is exactly one point common to ℓ_1 and ℓ_2, as claimed. □

Theorem 1.7.4

Theorem 1.7.4. Two lines have exactly one point in common.

Proof. Let ℓ_1 and ℓ_2 be lines. By A.3, there is a point P which lies on both ℓ_1 and ℓ_2. By Theorem 1.7.2, there is at most one point which lies on both ℓ_1 and ℓ_2. Therefore, there is exactly one point common to ℓ_1 and ℓ_2, as claimed. □

Theorem 1.7.5

Theorem 1.7.5. If P is any point, there is at least one line which does not pass through P.

Proof. By A.6, there exists a line ℓ. If this line does not pass through P, then we are done. So without loss of generality, we can assume that ℓ passes through P.

Theorem 1.7.5

Theorem 1.7.5. If P is any point, there is at least one line which does not pass through P.

Proof. By A.6, there exists a line ℓ. If this line does not pass through P, then we are done. So without loss of generality, we can assume that ℓ passes through P. By A.4, line ℓ contains at least three points, so there is another point P' on line ℓ. By A.5, there is at least one point P'' which does not lie on ℓ. By Theorem 1.7.3, there is a unique line ℓ' which contains P' and P''. Notice that ℓ and ℓ' are different lines, since P'' lies on ℓ' but P'' does not lie on ℓ.

Theorem 1.7.5

Theorem 1.7.5. If P is any point, there is at least one line which does not pass through P.

Proof. By A.6, there exists a line ℓ. If this line does not pass through P, then we are done. So without loss of generality, we can assume that ℓ passes through P. By A.4, line ℓ contains at least three points, so there is another point P' on line ℓ. By A.5, there is at least one point P'' which does not lie on ℓ. By Theorem 1.7.3, there is a unique line ℓ' which contains P' and P''. Notice that ℓ and ℓ' are different lines, since P'' lies on ℓ' but P'' does not lie on ℓ. By Theorem 1.7.4, ℓ and ℓ' have exactly one point in common, so this point must be point P'. Since point P lies on ℓ, then point P cannot also lie on ℓ'. Therefore ℓ' is a line which does not contain point P, as claimed. □

Theorem 1.7.5

Theorem 1.7.5. If P is any point, there is at least one line which does not pass through P.

Proof. By A.6, there exists a line ℓ. If this line does not pass through P, then we are done. So without loss of generality, we can assume that ℓ passes through P. By A.4, line ℓ contains at least three points, so there is another point P' on line ℓ. By A.5, there is at least one point P'' which does not lie on ℓ. By Theorem 1.7.3, there is a unique line ℓ' which contains P' and P''. Notice that ℓ and ℓ' are different lines, since P'' lies on ℓ' but P'' does not lie on ℓ. By Theorem 1.7.4, ℓ and ℓ' have exactly one point in common, so this point must be point P'. Since point P lies on ℓ, then point P cannot also lie on ℓ'. Therefore ℓ' is a line which does not contain point P, as claimed. □

Theorem 1.7.6

Theorem 1.7.6. Every point lies on at least three lines.

Proof. Let P be an arbitrary point, which is known to exist by Theorem 1.7.1. By Theorem 1.7.5, there is at least one line ℓ which does not pass through point P. By A.4, line ℓ contains at least three points, say P_1, P_2, and P_3 (notice that P is distinct from P_1, P_2, and P_3).

Theorem 1.7.6

Theorem 1.7.6. Every point lies on at least three lines.

Proof. Let P be an arbitrary point, which is known to exist by Theorem 1.7.1. By Theorem 1.7.5, there is at least one line ℓ which does not pass through point P. By A.4, line ℓ contains at least three points, say P_1, P_2, and P_3 (notice that P is distinct from P_1, P_2, and P_3). By Theorem 1.7.3, each of these points determines a unique line which also contains point P, say line ℓ_1, ℓ_2, and ℓ_3, respectively. Notice that the lines ℓ_1, ℓ_2, and ℓ_3 are distinct, for if two of the lines coincided then the common line would share two points with line ℓ (for example, if ℓ_1 and ℓ_2 are the same line then this line shares the points P_1 and P_2 with line ℓ), contradicting Theorem 1.7.4. So the three lines ℓ_1, ℓ_2, and ℓ_3 are distinct lines containing point P, as claimed. □

Theorem 1.7.6

Theorem 1.7.6. Every point lies on at least three lines.

Proof. Let P be an arbitrary point, which is known to exist by Theorem 1.7.1. By Theorem 1.7.5, there is at least one line ℓ which does not pass through point P. By A.4, line ℓ contains at least three points, say P_1, P_2, and P_3 (notice that P is distinct from P_1, P_2, and P_3). By Theorem 1.7.3, each of these points determines a unique line which also contains point P, say line ℓ_1, ℓ_2, and ℓ_3, respectively. Notice that the lines ℓ_1, ℓ_2, and ℓ_3 are distinct, for if two of the lines coincided then the common line would share two points with line ℓ (for example, if ℓ_1 and ℓ_2 are the same line then this line shares the points P_1 and P_2 with line ℓ), contradicting Theorem 1.7.4. So the three lines ℓ_1, ℓ_2, and ℓ_3 are distinct lines containing point P, as claimed. □

Theorem 1.7.7

Theorem 1.7.7. If there exists one line which contains exactly *n* points, then every line contains exactly *n* points.

Proof. Let ℓ be a line containing exactly n points, $P_1, P_2, \ldots, P_n$. Let ℓ' be a line other than line ℓ (which exists by Theorems 1.7.1 and 1.7.6). By Theorem 1.7.4, ℓ and ℓ' have exactly one point in common; we take this point to be P_1, without loss of generality. By A.4, ℓ' contains some point P_2' distinct from P_1. Notice that P_2' is distinct from $P_2, P_3, \ldots, P_n$ by Theorem 1.7.4. See Figure 1.6 below.

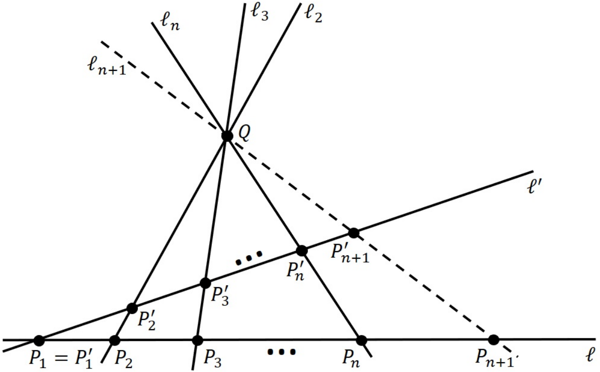
Theorem 1.7.7

Theorem 1.7.7. If there exists one line which contains exactly n points, then every line contains exactly n points.

Proof. Let ℓ be a line containing exactly n points, $P_1, P_2, \ldots, P_n$. Let ℓ' be a line other than line ℓ (which exists by Theorems 1.7.1 and 1.7.6). By Theorem 1.7.4, ℓ and ℓ' have exactly one point in common; we take this point to be P_1, without loss of generality. By A.4, ℓ' contains some point P_2' distinct from P_1. Notice that P_2' is distinct from $P_2, P_3, \ldots, P_n$ by Theorem 1.7.4. See Figure 1.6 below.

Theorem 1.7.7 (continued 1)

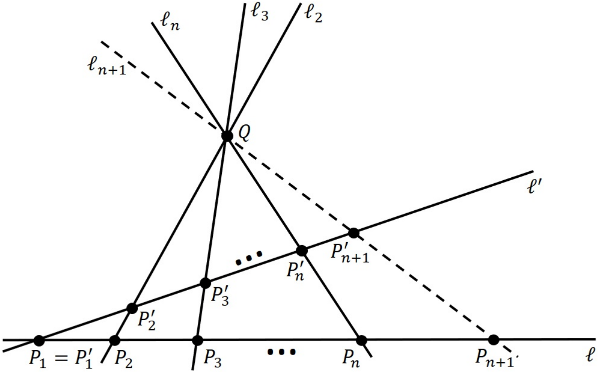
Proof (continued).



By Theorem 1.7.3, there is a unique line, say ℓ_2, containing both P_2 and P_2'. By A.4, there is a third point Q, distinct from P_2 and P_2', belonging to ℓ_2. By Theorem 1.7.4, point Q is distinct from $P_1, P_2, \ldots, P_n$ (consider lines ℓ and ℓ_2). By Theorem 1.7.3, Q determined a unique line with each of the points $P_3, P_4, \ldots, P_n$, say $\ell_3, \ell_4, \ldots, \ell_n$, respectively.

Theorem 1.7.7 (continued 2)

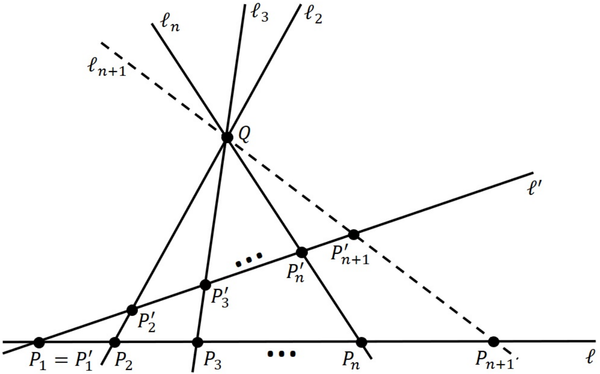
Proof (continued).



By Theorem 1.7.4, line $\ell_3, \ell_4, \ldots, \ell_n$ are distinct and are distinct from ℓ. Also by Theorem 1.7.4, lines $\ell_3, \ell_4, \ldots, \ell_n$ intersect ℓ' in unique points $P'_3, P'_4, \ldots, P'_n$, respectively, distinct and also distinct from $P'_1 = P_1$ and P'_2 by Theorem 1.7.2. Hence line ℓ' contains at least n points.

Theorem 1.7.7 (continued 3)

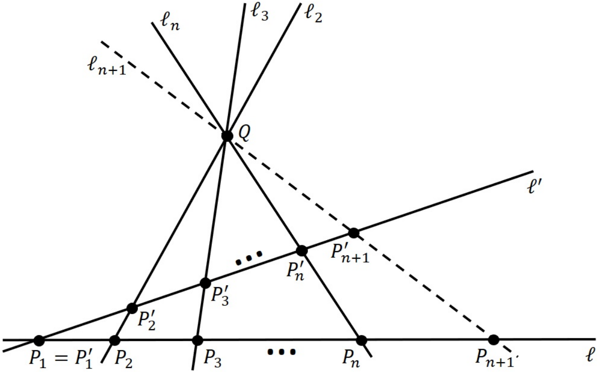
Proof (continued).



We now show that ℓ' contains no more than n points. ASSUME to the contrary that ℓ' contains another point, say P'_{n+1}. By Theorem 1.7.3 there is a unique line ℓ_{n+1} containing Q and P'_{n+1} and, again, by Theorem 1.7.4 this line is distinct from $\ell_1, \ell_2, \ldots, \ell_n$ and distinct from ℓ.

Theorem 1.7.7 (continued 4)

Proof (continued).



By Theorem 1.7.4, there is a unique point common to ℓ_{n+1} and ℓ, which we denote P_{n+1}, and which is distinct from $P_1, P_2, \ldots, P_n$ by Theorem 1.7.3. But then line ℓ has $n+1$ points, a CONTRADICTION. So the assumption that ℓ' has more than n points is false. Since ℓ' is an arbitrary line distinct from line ℓ, the claim follows. □

Theorem 1.7.8

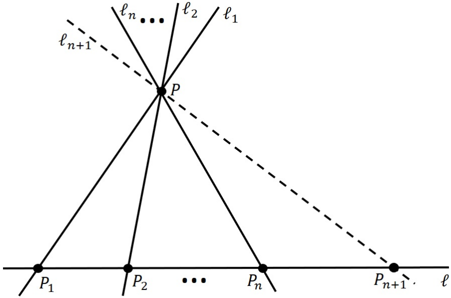
Theorem 1.7.8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Proof. Let P be an arbitrary point. By Theorem 1.7.5 there is at least one line ℓ which does not pass through P. By Theorem 1.7.7, ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.3, P and each of $P_1, P_2, \ldots, P_n$ determines a line $\ell_1, \ell_2, \ldots, \ell_n$. and these lines are distinct by Theorem 1.7.4. Therefore there are at least n lines passing through P.

Theorem 1.7.8

Theorem 1.7.8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Proof. Let P be an arbitrary point. By Theorem 1.7.5 there is at least one line ℓ which does not pass through P. By Theorem 1.7.7, ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.3, P and each of $P_1, P_2, \ldots, P_n$ determines a line $\ell_1, \ell_2, \ldots, \ell_n$. and these lines are distinct by Theorem 1.7.4.

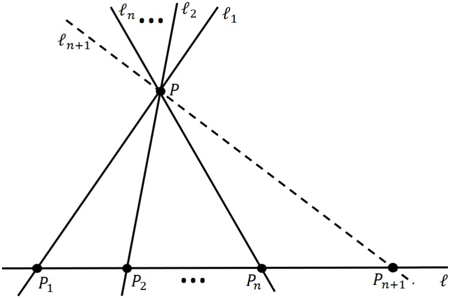


Therefore there are at least n lines passing through P.

Theorem 1.7.8

Theorem 1.7.8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Proof. Let P be an arbitrary point. By Theorem 1.7.5 there is at least one line ℓ which does not pass through P. By Theorem 1.7.7, ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.3, P and each of $P_1, P_2, \ldots, P_n$ determines a line $\ell_1, \ell_2, \ldots, \ell_n$. and these lines are distinct by Theorem 1.7.4.

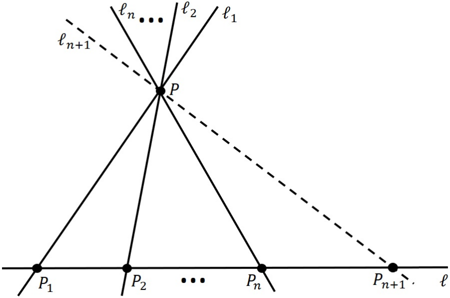


Therefore there are at least n lines passing through P.

Theorem 1.7.8 (continued)

Theorem 1.7.8. If there exists one line which contains exactly n points, then exactly n lines pass through every point.

Proof (continued). Next, ASSUME there is at least one additional line, ℓ_{n+1}, passing through P. By Theorem 1.7.4, ℓ_{n+1} must intersect ℓ is a unique point, say P_{n+1}, so that P_{n+1} is distinct from $P_1, P_2, \ldots, P_n$. But then ℓ contains $n+1$ points, a CONTRADICTION. So the assumption that there are more than n lines passing through P is false, and hence there are exactly n line through point P. Since P is an arbitrary point, the claim follows. □



Theorem 1.7.9

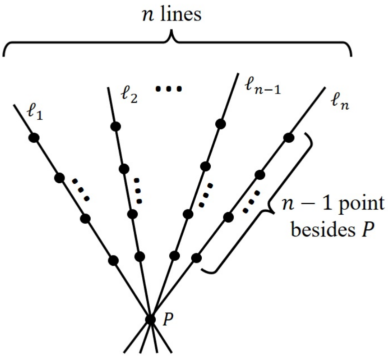
Theorem 1.7.9. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ points.

Proof. By Theorem 1.7.1 there exists at least one point P and by Theorem 1.7.8 there are exactly n lines, $\ell_1, \ell_2, \ldots, \ell_n$ passing through P. By Theorem 1.7.3 (two points determine exactly one line), every point in the system, except point P itself, lies on exactly one line passing through P; so if we count all the distinct points on lines $\ell_1, \ell_2, \ldots, \ell_n$ then we have the total number of points. By Theorem 1.7.7 every line contains exactly n points. So each of $\ell_1, \ell_2, \ldots, \ell_n$ contains $n - 1$ points besides point P. Therefore, there are a total of $n(n-1) + 1 = n^2 - n + 1$ points, as claimed. □

Theorem 1.7.9

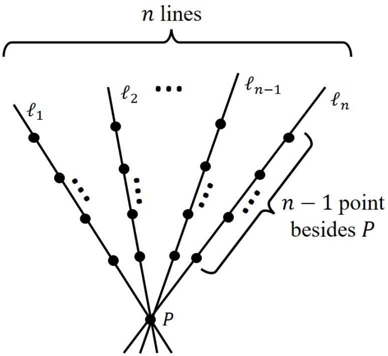
Theorem 1.7.9. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ points.

Proof. By Theorem 1.7.1 there exists at least one point P and by Theorem 1.7.8 there are exactly n lines, $\ell_1, \ell_2, \ldots, \ell_n$ passing through P. By Theorem 1.7.3 (two points determine exactly one line), every point in the system, except point P itself, lies on exactly one line passing through P; so if we count all the distinct points on lines $\ell_1, \ell_2, \ldots, \ell_n$ then we have the total number of points. By Theorem 1.7.7 every line contains exactly n points. So each of $\ell_1, \ell_2, \ldots, \ell_n$ contains $n - 1$ points besides point P. Therefore, there are a total of $n(n-1) + 1 = n^2 - n + 1$ points, as claimed. □



Theorem 1.7.9

Theorem 1.7.9. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ points.



Proof. By Theorem 1.7.1 there exists at least one point P and by Theorem 1.7.8 there are exactly n lines, $\ell_1, \ell_2, \ldots, \ell_n$ passing through P. By Theorem 1.7.3 (two points determine exactly one line), every point in the system, except point P itself, lies on exactly one line passing through P; so if we count all the distinct points on lines $\ell_1, \ell_2, \ldots, \ell_n$ then we have the total number of points. By Theorem 1.7.7 every line contains exactly n points. So each of $\ell_1, \ell_2, \ldots, \ell_n$ contains $n-1$ points besides point P. Therefore, there are a total of $n(n-1) + 1 = n^2 - n + 1$ points, as claimed. □

Theorem 1.7.10

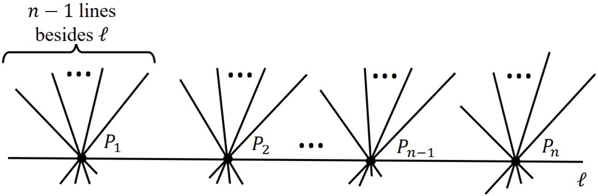
Theorem 1.7.10. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ lines.

Proof. By A.6 there exists at least one line ℓ, and by Theorem 1.7.7 line ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.4 (two lines have exactly one point in common), every line in the system, except ℓ itself, passes through exactly one of the points $P_1, P_2, \ldots, P_n$.

Theorem 1.7.10

Theorem 1.7.10. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ lines.

Proof. By A.6 there exists at least one line ℓ, and by Theorem 1.7.7 line ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.4 (two lines have exactly one point in common), every line in the system, except ℓ itself, passes through exactly one of the points $P_1, P_2, \ldots, P_n$.

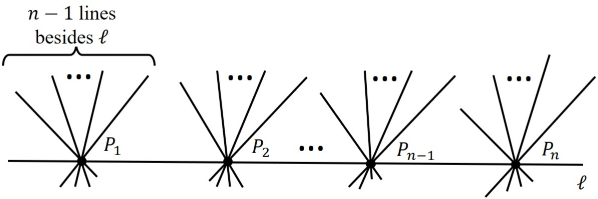


By Theorem 1.7.8 exactly n lines (including line ℓ) pass through each of the points $P_1, P_2, \ldots, P_n$. So there is a total of $n(n-1) + 1 = n^2 - n + 1$ lines, as claimed. □

Theorem 1.7.10

Theorem 1.7.10. If there exists one line which contains exactly n points, then the system contains exactly $n^2 - n + 1$ lines.

Proof. By A.6 there exists at least one line ℓ, and by Theorem 1.7.7 line ℓ contains exactly n points, say $P_1, P_2, \ldots, P_n$. By Theorem 1.7.4 (two lines have exactly one point in common), every line in the system, except ℓ itself, passes through exactly one of the points $P_1, P_2, \ldots, P_n$.



By Theorem 1.7.8 exactly n lines (including line ℓ) pass through each of the points $P_1, P_2, \ldots, P_n$. So there is a total of $n(n-1)+1 = n^2 - n + 1$ lines, as claimed. □