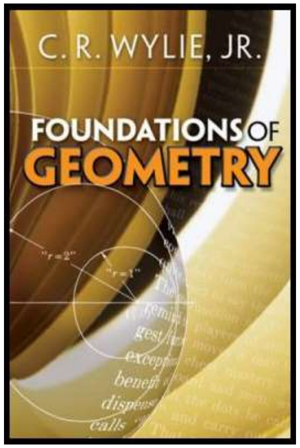
Foundations of Geometry

Chapter 2. Euclidean Geometry

2.4. The Measurements of Distance—Proofs of Theorems



Theorem 2.4.1

Theorem 2.4.1. If P, Q, and R are points such that for some unit pair, $\alpha = \{A, A'\}$, $m_\alpha(P, Q) + m_\alpha(Q, R) = m_\alpha(P, R)$ then for any other unit pair $\beta = \{B, B'\}$, we have $m_\beta(P, Q) + m_\beta(Q, R) = m_\beta(P, R)$.

Proof. By Postulate 9 we can relate the distances between points relative to the different unit pairs as

$$m_\alpha(P, Q) = m_\alpha(B, B')m_\beta(P, Q),$$
$$m_\alpha(Q, E) = m_\alpha(B, B')m_\beta(Q, R),$$
$$m_\alpha(P, R) = m_\alpha(B, B')m_\beta(P, R).$$

Substituting these relationships into $m_\alpha(P, Q) + m_\alpha(Q, R) = m_\alpha(P, R)$ gives: $m_\alpha(B, B')m_\beta(P, Q) + m_\alpha(B, B')m_\beta(Q, R) = m_\alpha(B, B')m_\beta(P, R)$. Now $m_\alpha(B, B') \neq 0$ since, by the definition of "unit pair," $B \neq B'$. So dividing through by $m_\alpha(B, B')$, the previous equation yields

$$m_\beta(P, Q) + m_\beta(Q, R) = m_\beta(P, R),$$

as claimed. □