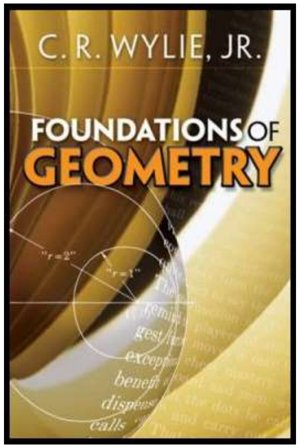
Foundations of Geometry

Chapter 2. Euclidean Geometry

2.5. Order Relations—Proofs of Theorems



Theorem 2.5.1

Theorem 2.5.1. Let A, B, and C be three points on line ℓ and let x, y, and z be, respectively. the coordinates of these points in a coordinate system on ℓ. Then B is between A and C if and only if y is between x and z.

Proof. First, suppose that y is between x and z. Then either $x > y > z$ or $x < y < z$. If $x > y > z$ the we have $x - y > 0$, $y - z > 0$, and $x - z > 0$, so that in terms of absolute values we have $|x - y| = x - y$, $|y - z| = y - z$, and $|x - z| = x - z$. By Postulate 11 (The Ruler Postulate), $|x - y| = AB$, $|y - z| = BC$, and $|x - z| = AC$. Substituting we have

$$AB + BC = |x - x| + |y - z| = (x - y) + (y - z) = x - z = |x - z| = AC.$$

So by Definition 2.5.1, B is between A and C, as claimed. If $x < y < z$, then the argument is the same except that the absolute values are the negatives of those given above.

Theorem 2.5.1 (continued 1)

Theorem 2.5.1. Let A, B, and C be three points on line ℓ and let x, y, and z be, respectively. the coordinates of these points in a coordinate system on ℓ. Then B is between A and C if and only if y is between x and z.

Proof (continued). Now suppose that B is between A and C. Then by Definition 2.5.1, $AB + BC = AC$ or, in terms of coordinates, $|x - y| + |y - z| = |x - z|$. Since x, y, and z are distinct real numbers, there are six possible order relations:

$$y > x > z, \quad x > y > z, \quad x > z > y,$$
$$z > x > y, \quad z > y > x, \quad y > z > z.$$

We simply exhaustively check these six cases. First, if $y > x > z$ then $AB = |x - y| = y - x$, $AC = |x - z| = x - z$, and $BC = |y - z| = y - z$. Substituting into $AB + BC = AC$ we get $(y - x) + (y - z) = x - z$ or $2y = 2z$ or $x = y$. But this cannot be the case since A and B are distinct points and so the relation $y > x > z$ is not possible.

Theorem 2.5.1 (continued 2)

Theorem 2.5.1. Let A, B, and C be three points on line ℓ and let x, y, and z be, respectively. the coordinates of these points in a coordinate system on ℓ. Then B is between A and C if and only if y is between x and z.

Proof (continued). Similarly, the relations $z > x > y$, $y > x > z$, and $y > z > x$ are not possible. However, the relations $x > y > z$ and $z > y > x$ are possible. For example, if $x > y > z$ then $AB = |x - y| = x - y$, $AC = |x - z| = x - z$, and $BC = |y - z| = y - z$. Substituting into $AB + BC = AC$ we get $(x - y) + (y - z) = x - z$ or $x - z = x - z$, which is possible! So if B is between A and C then either $x > y > z$ of $z > y > x$; that is, y is between x and z, as claimed. □

Theorem 2.5.3

Theorem 2.5.3. Let A and B be distinct points and let a and b be, respectively, the coordinates of these points in any coordinate system on $\overleftrightarrow{AB}$. Then if $a < b$, the ray $\overrightarrow{AB}$ is the same as the set of points whose coordinates x satisfy the condition $a \leq x$. If $a > b$, the ray $\overrightarrow{AB}$ is the same as the set of points whose coordinates satisfy the condition $a \geq x$.

Proof. First, suppose that $a < b$. If X is any point of ray $\overrightarrow{AB}$, then X is either a point of the segment $\overline{AB}$ (so that X is between A and B) or else is a point such that B is between A an X. If X is a point of $\overline{AB}$, then by Theorem 2.5.2 the coordinate x of point X must be such that $a \leq x \leq b$ (with equality when X is an endpoint of $\overline{AB}$). Next, if B lies between A and X, then again by Theorem 2.5.2 we have $a < b < x$. In either case, $a \leq x$ as claimed.
Second, suppose $a \leq x$. Then either $a \leq x \leq b$ or $a < b < x$. Hence X either belongs to the segment $\overline{AB}$ or is a point such that B lies between A and X, respectively. In both cases, X belongs to $\overrightarrow{AB}$ by Definition 2.5.3, as claimed. □

Theorem 2.5.5. The Point-Plotting Theorem

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If $\overrightarrow{AB}$ is a ray and d a positive number, then there is exactly one point on $\overrightarrow{AB}$ and one point on the ray opposite to $\overrightarrow{AB}$ such that the distance from A to each of these points relative to a given unit pair is d.

Proof. By Postulate 10 there is a point U on line $\overleftrightarrow{AB}$ such that $AU = 1$ relative to the given unit pair. By Postulate 11 (The Ruler Postulate) there is a coordinate system on $\overleftrightarrow{AB}$ in which point A has coordinate 0 and point U has coordinate 1. Then in this coordinate system, there is a unique point D whose coordinate is the given positive number d and a unique point D' whose coordinate is the negative number $-d$. By Postulate 11 (again) we have $AD = |0 - d| = d$ and $AD' = |0 - (-d)| = d$. Hence the distance from A to each of the points D and D' is d.

Theorem 2.5.5. The Point-Plotting Theorem (continued)

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If $\overrightarrow{AB}$ is a ray and d a positive number, then there is exactly one point on $\overrightarrow{AB}$ and one point on the ray opposite to $\overrightarrow{AB}$ such that the distance from A to each of these points relative to a given unit pair is d.

Proof (continued). By Theorem 2.5.4, the points D and D' are on opposite rays on the line $\overleftrightarrow{AB}$, and so one of them is on ray $\overrightarrow{AB}$ and one is on the opposite ray, as claimed. □

Theorem 2.5.7

Theorem 2.5.7. The intersection of two convex sets is a convex set.

Proof. Let S_1 and S_2 be two convex sets and let $\overline{AB}$ be any segment whose endpoints lie in the intersection of S_1 and S_2, $S_1 \cap S_2$. From the definition of intersection of two sets (Definition 2.3.3), endpoints A and B of the segment lie in both set S_1 and set S_2. Since both S_1 and S_2 are convex by hypothesis, then the segment $\overline{AB}$ lies entirely in both S_1 and S_2 by the definition of convex (Definition 2.5.4). Therefore $\overline{AB}$ lies entirely in the intersection of S_1 and S_2. Since segment $\overline{AB}$ was an arbitrary segment whose endpoints line in $S_1 \cap S_2$, then the intersection is convex, as claimed. □

Theorem 2.5.9

Theorem 2.5.9. The points of space which do not lie in a given plane form two sets such that:

(1) each set is convex, and

(2) any segment joining a point in one set to a point in the other set intersects the given plane.

Proof. Let π be an arbitrary plane and let O be an arbitrary point which does not lie in π. Now any given segment with O as an endpoint either intersects π or not. So every point in space which is not a point of π must belong to one or the other side of the following two nonempty sets:

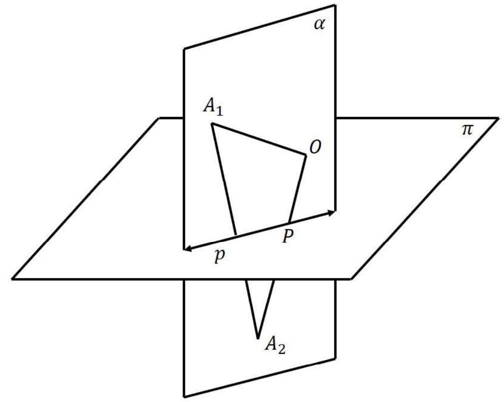
(1) the set S_1 consisting of O and all points A_1 such that the segment $\overline{OA_1}$ does not intersect π, or

(2) the set S_2 consisting of all points A_2 not in π such that the segment $\overline{OA_2}$ intersects π.

We next show that the sets S_1 and S_2 have the properties claimed in the theorem.

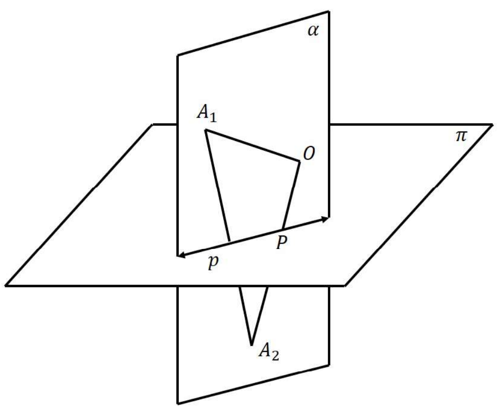
Theorem 2.5.9 (continued 1)

Proof (continued). First, suppose that A_1 and A_2 are arbitrary points in S_1 and S_2, respectively. By the definition of S_2, segment $\overline{OA_2}$ intersects intersects π at some point P. Therefore, if $A_1 = O$ then $\overline{A_1A_2} = \overline{OA_2}$ intersects π, as claimed. So we can assume without loss of generality that $A_1 \neq O$. Let α be a plane containing points A_1, A_2, and O (there are multiple such planes if A_1, A_2, O are collinear). Since P is a point on $\overline{OA_2}$ then by Postulate 5 point P is in plane α and so the two planes α and π intersect. By Postulate 6 the planes intersect in some line p. Now $\overline{OA_1}$ does not intersect plane π (by the choice of A_1 and the definition of S_1).



Theorem 2.5.9 (continued 2)

Proof (continued). So $\overline{OA_1}$ does not intersect line p since it lies in plane π. Hence in plane α the points O and A_1 lie on the same side of line p by Postulate 12 (The Plane -Separation Postulate). Since $\overline{OA_2}$ intersects plane π and must do so at a point on line p, then O and A_2 lie on opposite sides of p (also by Postulate 12). Thus A_1 and A_2 are on opposite sides of p and so by Postulate 12, the segment $\overline{A_1A_2}$ intersects line p and hence plane π, as claimed. We now turn out attention to the claims that S_1 and S_2 are convex.



Theorem 2.5.9 (continued 3)

Theorem 2.5.9. The points of space which do not lie in a given plane form two sets such that:

(1) each set is convex, and

(2) any segment joining a point in one set to a point in the other set intersects the given plane.

Proof (continued). For the convexity of S_1, let A_1 and A_1' by two arbitrary points in S_1. ASSUME segment $\overline{A_1A_1'}$ does not lie entirely in S_1. Then there must be at least one point Q of $\overline{A_1A_1'}$ (that is, a point between A_1 and A_1' on the segment) which is either a point of S_2 or a point of π. First, if Q is a point of S_2 then as shown above, both segment $\overline{A_1Q}$ and segment $\overline{A_1'Q}$ intersect plane π. These points of intersection must be distinct since one lies on the ray $\overrightarrow{QA_1}$ and the other lines on the opposite ray $\overrightarrow{QA_1'}$. But then the line $\overleftrightarrow{A_1A_1'}$ intersects plane π in two points and so by Postulate 5 $\overleftrightarrow{A_1A_1'}$ lies in plane π. But then A_1 and A_1' themselves line in π, a (first) CONTRADICTION to the fact that A_1 and A_1' are in S_1.

Theorem 2.5.9 (continued 4)

Proof (continued). Second, if Q is a point of π then a plane α containing points A_1, A_1', O intersects plane π at point Q (remember, Q is a point of $\overline{A_1A_1'}$). So planes π and α intersect in some line p by Postulate 6. Since A_1, A_1', and O line in plane α and O is between A_1 and A_1' on $\overline{A_1A_1'}$, then by Postulate 12 (The Plane-Separation Postulate) A_1 and A_1' are on opposite sides of p in plane α. Now point O is also in plane α and not on line p, so either O and A_1 are on opposite sides of p, or O and A_1' are on opposite sides of p. So either $\overline{OA_1}$ or $\overline{OA_1'}$ intersects line p and hence intersects plane π, but this is a (second) CONTRADICTION to the fact that A_1 and A_1' are in S_1. So the assumption that segment $\overline{A_1A_1'}$ does not lie entirely in S_1 is false and hence every point of $\overline{A_1A_1'}$ must be a point of S_1. Since A_1 and A_1' are arbitrary points of S_1, we have that S_1 is a convex set, as claimed. "By an almost identical argument" (as Wylie states on page 66) we can show that S_2 is also convex, as claimed. □

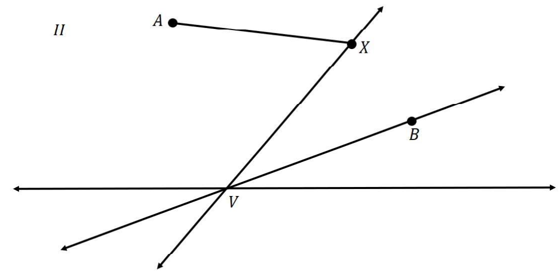
Theorem 2.5.10

Theorem 2.5.10. If V is any point on the edge of a halfplane H and if A, B, and X are three points in the union of H and its edge such that:

(1) no two of the points A, B, X are collinear with V and

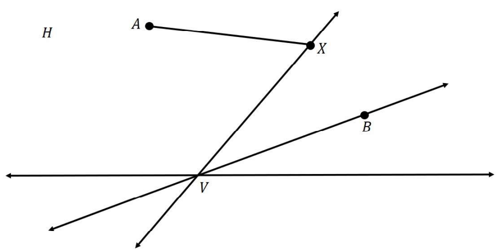
(2) A and B lie on opposite sides of $\overleftrightarrow{VX}$,

then A and X lie on the same side of $\overleftrightarrow{VB}$, and B and X lie on the same side of $\overleftrightarrow{VA}$.



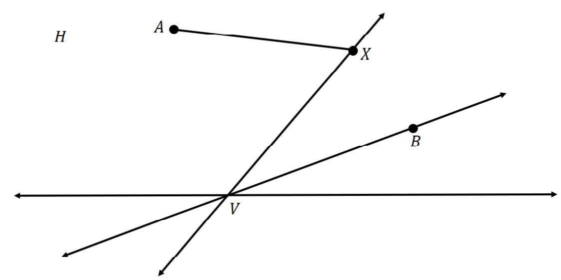
Proof. Since points A, B, X lie in the union of H and its edge, and since this set is convex by Exercise 2.5.A, then $\overline{AX}$ lies in the union of H and its edge. Since V is on the edge of H, and B is in H, then ray $\overrightarrow{VB}$ lies in the union of H and its edge by Exercise 2.5.B.

Theorem 2.5.10 (continued 1)



Proof (continued). If $\overline{AX}$ and $\overleftrightarrow{VB}$ intersect, then their intersection must be a point on $\overrightarrow{VB}$ (and not on the ray opposite to $\overrightarrow{VB}$ since the only point of this ray in the union of H and its edge is V by Exercise 2.5.C); notice that we do not have A, X, B collinear so the point of intersection cannot by V. But we have that points A and B are on opposite sides of $\overleftrightarrow{VX}$ by hypothesis. Hence, with the exception of the distinct points V and X, $\overline{AX}$ and $\overrightarrow{VB}$ lie on opposite sides of $\overleftrightarrow{VX}$ by Exercise 2.5.C, and therefore can have no point in common.

Theorem 2.5.10 (continued 2)



Proof (continued). Thus, since $\overline{AX}$ can intersect neither the ray $\overrightarrow{VB}$ nor the ray opposite to $\overrightarrow{VB}$, then points A and X are on the same side of $\overleftrightarrow{VB}$. Finally, an identical argument shows that B and X lie on the same side of $\overleftrightarrow{VA}$, as asserted. □