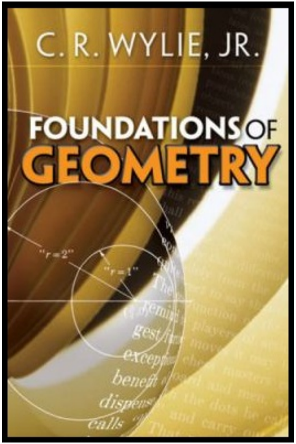
# Foundations of Geometry

**Chapter 2. Euclidean Geometry**

2.7. Further Properties of Angles—Proofs of Theorems



# Table of contents

# Theorem 2.7.1

**Theorem 2.7.1.** The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

**Proof.** Let $S_1$ be the interior of $\angle AVB$; that is, $S_1 = \{X \mid \overrightarrow{VX}$ lies between $\overrightarrow{VA}$ and $\overrightarrow{VB}\}$, and let $S_2$ be the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

# Theorem 2.7.1

**Theorem 2.7.1.** The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

**Proof.** Let $S_1$ be the interior of $\angle AVB$; that is, $S_1 = \{X \mid \overrightarrow{VX}$ lies between $\overrightarrow{VA}$ and $\overrightarrow{VB}\}$, and let $S_2$ be the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

Let $X \in S_1$, so that $\overrightarrow{VX}$ lies between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Then by the Definition 2.6.2, there is a halfplane $H$ such that:

(1) the edge of the halfplane $H$ contains $V$,

(2) the points $Z, B, X$ lie in the union of $H$ and its edge,

(3) no two of the points $Z, B, X$ are collinear with $V$, and

(4) $m\angle AVX + m\angle XVB = m\angle AVB$.

# Theorem 2.7.1

**Theorem 2.7.1.** The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

**Proof.** Let $S_1$ be the interior of $\angle AVB$; that is, $S_1 = \{X \mid \overrightarrow{VX}$ lies between $\overrightarrow{VA}$ and $\overrightarrow{VB}\}$, and let $S_2$ be the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

Let $X \in S_1$, so that $\overrightarrow{VX}$ lies between $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Then by the Definition 2.6.2, there is a halfplane $H$ such that:

(1) the edge of the halfplane $H$ contains $V$,

(2) the points $Z, B, X$ lie in the union of $H$ and its edge,

(3) no two of the points $Z, B, X$ are collinear with $V$, and

(4) $m\angle AVX + m\angle XVB = m\angle AVB$.

# Theorem 2.7.1 (continued 1)

**Proof (continued).** ASSUME that points $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$. Then by the Protractor Postulate (Postulate 15), there exists a one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VX}$ the number 0 and assigns to $\overrightarrow{VA}$ and $\overrightarrow{VB}$ numbers $a > 0$ and $b > 0$, respectively, such that $m\angle AVX = |a - x| = a$, $m\angle XVB = |b - 0| = b$, and $m\angle AVB = |b - a|$. Then by (4) from above, we have $a + b = |b - a|$, a CONTRADICTION since this holds for no positive $a$ and $b$. Therefore the assumption that $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$ is false, and hence so $A$ and $B$ lie on opposite sides of $\overleftrightarrow{VX}$.

# Theorem 2.7.1 (continued 1)

**Proof (continued).** ASSUME that points $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$. Then by the Protractor Postulate (Postulate 15), there exists a one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VX}$ the number 0 and assigns to $\overrightarrow{VA}$ and $\overrightarrow{VB}$ numbers $a > 0$ and $b > 0$, respectively, such that $m\angle AVX = |a - x| = a$, $m\angle XVB = |b - 0| = b$, and $m\angle AVB = |b - a|$. Then by (4) from above, we have $a + b = |b - a|$, a CONTRADICTION since this holds for no positive $a$ and $b$. Therefore the assumption that $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$ is false, and hence so $A$ and $B$ lie on opposite sides of $\overleftrightarrow{VX}$. Now by Theorem 2.5.10, $A$ and $X$ lie on the same side of $\overleftrightarrow{VB}$, and $B$ and $X$ lie on the same side of $\overleftrightarrow{VA}$. That is, $X$ is in both the halfplane determined by $\overleftrightarrow{VA}$ and $B$, and in the halfplane determined by $\overleftrightarrow{VB}$ and $A$. So $X \in S_2$ and hence $S_1 \subset S_2$.

# Theorem 2.7.1 (continued 1)

**Proof (continued).** ASSUME that points $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$. Then by the Protractor Postulate (Postulate 15), there exists a one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VX}$ the number 0 and assigns to $\overrightarrow{VA}$ and $\overrightarrow{VB}$ numbers $a > 0$ and $b > 0$, respectively, such that $m\angle AVX = |a - x| = a$, $m\angle XVB = |b - 0| = b$, and $m\angle AVB = |b - a|$. Then by (4) from above, we have $a + b = |b - a|$, a CONTRADICTION since this holds for no positive $a$ and $b$. Therefore the assumption that $A$ and $B$ are on the same side of $\overleftrightarrow{VX}$ is false, and hence so $A$ and $B$ lie on opposite sides of $\overleftrightarrow{VX}$. Now by Theorem 2.5.10, $A$ and $X$ lie on the same side of $\overleftrightarrow{VB}$, and $B$ and $X$ lie on the same side of $\overleftrightarrow{VA}$. That is, $X$ is in both the halfplane determined by $\overleftrightarrow{VA}$ and $B$, and in the halfplane determined by $\overleftrightarrow{VB}$ and $A$. So $X \in S_2$ and hence $S_1 \subset S_2$.

# Theorem 2.7.1 (continued 2)

**Proof (continued).** Next, suppose $X \in S_2$; that is, $X$ is in the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$. In this case $X$ lies on the same side of $\overleftrightarrow{VA}$ as $B$, and on the same side of $\overleftrightarrow{VB}$ as $A$. So by the Protractor Postulate (Postulate 15), there is a first one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VA}$ the number 0, assigns to $\overrightarrow{VB}$ the number $b$, and assigns to $\overrightarrow{VX}$ the number $x$ such that $m\angle AVB = |b - 0| = b$, $m\angle AVX = |x - 0| = x$, and $m\angle BVX = |b - x|$ (see Figure 2.16(a)). Also by the Protractor Postulate (Postulate 15), there is a second one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VA}$ the number $a'$, assigns to $\overrightarrow{VB}$ the number 0, and assigns to $\overrightarrow{VX}$ the number $x'$ such that $m\angle AVB = |a' - 0| = a'$, $m\angle AVX = |a' - x'|$, and $m\angle BVX = |x' - 0| = x'$ (see Figure 2.16(b)).

# Theorem 2.7.1 (continued 2)

**Proof (continued).** Next, suppose $X \in S_2$; that is, $X$ is in the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$. In this case $X$ lies on the same side of $\overleftrightarrow{VA}$ as $B$, and on the same side of $\overleftrightarrow{VB}$ as $A$. So by the Protractor Postulate (Postulate 15), there is a first one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VA}$ the number 0, assigns to $\overrightarrow{VB}$ the number $b$, and assigns to $\overrightarrow{VX}$ the number $x$ such that $m\angle AVB = |b - 0| = b$, $m\angle AVX = |x - 0| = x$, and $m\angle BVX = |b - x|$ (see Figure 2.16(a)). Also by the Protractor Postulate (Postulate 15), there is a second one-to-one correspondence between the interval $[0, R]$ and the rays in a halfplane and its boundary which assigns to $\overrightarrow{VA}$ the number $a'$, assigns to $\overrightarrow{VB}$ the number 0, and assigns to $\overrightarrow{VX}$ the number $x'$ such that $m\angle AVB = |a' - 0| = a'$, $m\angle AVX = |a' - x'|$, and $m\angle BVX = |x' - 0| = x'$ (see Figure 2.16(b)).

# Theorem 2.7.1 (continued 3)

**Proof (continued).**

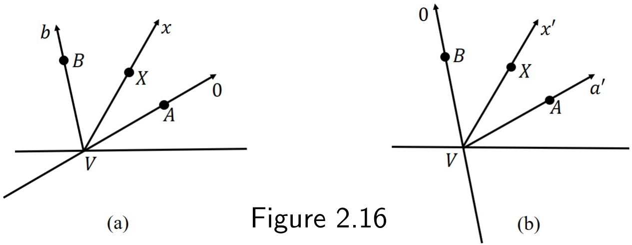



Figure 2.16

Of course, either $b < x$ or $b > x$. If $b < x$ then by the first correspondence $m\angle BVX = |b - x| = x - b$ and so $m\angle AVB + b\angle BVX = b + (x - b) = x = m\angle AVX$. But in the second correspondence, this relationship becomes $m\angle AVB + b\angle BVX = a' + x' = |a' - x'| = m\angle AVX$ and so we must have $a' + x' = |a' - x'|$. But this for no positive $a'$ and $x'$ and we cannot have $b < x$, so that we must have $b > x$.

# Theorem 2.7.1 (continued 4)

**Theorem 2.7.1.** The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

**Proof (continued).** With $b > x$ we have by the first correspondence that $m\angle BVX = |b - x| = b - x$. Also by the first correspondence, $m\angle AVX + m\angle XVB = x + (b - x) = b = m\angle AVB$ so, by Definition 2.6.2, ray $\overrightarrow{VX}$ lies between rays $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Therefore $X \in S_1$ and we have $S_2 \subset S_1$.

Therefore, $S_1 = S_2$. That is, the interior of $\angle AVB$ (set $S_1$) is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$ (set $S_2$), as claimed. □

# Theorem 2.7.1 (continued 4)

**Theorem 2.7.1.** The interior of $\angle AVB$ is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$.

**Proof (continued).** With $b > x$ we have by the first correspondence that $m\angle BVX = |b - x| = b - x$. Also by the first correspondence, $m\angle AVX + m\angle XVB = x + (b - x) = b = m\angle AVB$ so, by Definition 2.6.2, ray $\overrightarrow{VX}$ lies between rays $\overrightarrow{VA}$ and $\overrightarrow{VB}$. Therefore $X \in S_1$ and we have $S_2 \subset S_1$.

Therefore, $S_1 = S_2$. That is, the interior of $\angle AVB$ (set $S_1$) is the intersection of the halfplane determined by $\overleftrightarrow{VA}$ and $B$ and the halfplane determined by $\overleftrightarrow{VB}$ and $A$ (set $S_2$), as claimed. $\square$

# Theorem 2.7.2

**Theorem 2.7.2.** The interior of an angle is a convex set.

**Proof.** A halfplane is a convex set by definition (see the Plane-Separation Postulate, Postulate 12). By Theorem 2.7.1, the interior of an angle is the intersection of two halfplanes. By Theorem 2.5.7, the intersection of two convex sets is a convex set. Therefore, the interior of an angle is a convex set, as claimed. □

# Theorem 2.7.2

**Theorem 2.7.2.** The interior of an angle is a convex set.

**Proof.** A halfplane is a convex set by definition (see the Plane-Separation Postulate, Postulate 12). By Theorem 2.7.1, the interior of an angle is the intersection of two halfplanes. By Theorem 2.5.7, the intersection of two convex sets is a convex set. Therefore, the interior of an angle is a convex set, as claimed. □

# Theorem 2.7.3

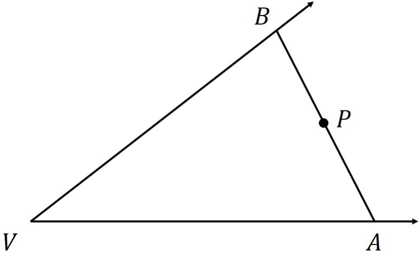
**Theorem 2.7.3.** If on each side of an angle a point other than the vertex is selected, every point between these points is in the interior of the angle.

**Proof.** Let $V$ be the vertex of the given angle. Let $A$ be a point on one side of the angle and let point $B$ a point on the other side of the angle (where $A$ and $B$ are distinct from $V$). Let $P$ be any point between $A$ and $B$ (see Figure 2.17).

# Theorem 2.7.3

**Theorem 2.7.3.** If on each side of an angle a point other than the vertex is selected, every point between these points is in the interior of the angle.

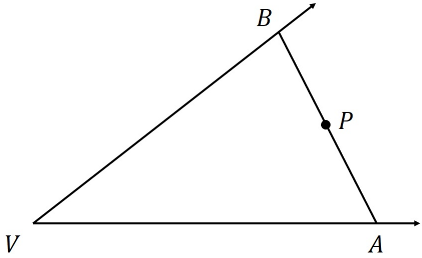
**Proof.** Let $V$ be the vertex of the given angle. Let $A$ be a point on one side of the angle and let point $B$ a point on the other side of the angle (where $A$ and $B$ are distinct from $V$). Let $P$ be any point between $A$ and $B$ (see Figure 2.17). For three collinear points, only one can be between the other two and since $P$ is between $A$ and $B$, then $B$ cannot be between $A$ and $P$. So the segment $\overline{AP}$ does not intersect the line $\overleftrightarrow{VB}$ (line $\overleftrightarrow{AP}$ intersects $\overleftrightarrow{VB}$ at point $B$, so there can be no second point of intersection of these two lines). So points $P$ and $A$ lie on the same side of $\overleftrightarrow{VB}$. Similarly, points $P$ and $B$ lie on the same side of $\overleftrightarrow{VA}$. Therefore, by Theorem 2.7.1, $P$ lies in the interior of the angle, as claimed. □



# Theorem 2.7.3

**Theorem 2.7.3.** If on each side of an angle a point other than the vertex is selected, every point between these points is in the interior of the angle.

**Proof.** Let $V$ be the vertex of the given angle. Let $A$ be a point on one side of the angle and let point $B$ a point on the other side of the angle (where $A$ and $B$ are distinct from $V$). Let $P$ be any point between $A$ and $B$ (see Figure 2.17). For three collinear points, only one can be between the




other two and since $P$ is between $A$ and $B$, then $B$ cannot be between $A$ and $P$. So the segment $\overline{AP}$ does not intersect the line $\overleftrightarrow{VB}$ (line $\overleftrightarrow{AP}$ intersects $\overleftrightarrow{VB}$ at point $B$, so there can be no second point of intersection of these two lines). So points $P$ and $A$ lie on the same side of $\overleftrightarrow{VB}$. Similarly, points $P$ and $B$ lie on the same side of $\overleftrightarrow{VA}$. Therefore, by Theorem 2.7.1, $P$ lies in the interior of the angle, as claimed. □

# Theorem 2.7.10

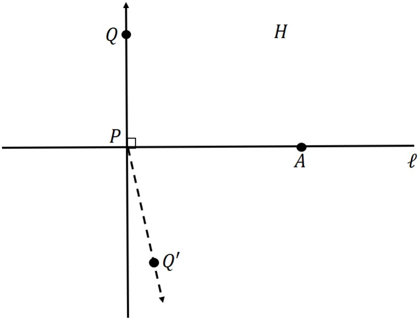
**Theorem 2.7.10.** At each point of a given line there is one and only one line which is perpendicular to the given line and lies in a given plane containing the line.

**Proof.** Let $P$ be a point on line $\ell$, let $\pi$ be a plane containing $\ell$, and let $A$ be a point on $\ell$ distinct from $P$. By the Plane-Separation Postulate (Postulate 12), line $\ell$ determines two halfplanes of $\pi$. Let $H$ be one of these halfplanes. By the Angle-Construction Theorem (Theorem 2.6.3), there is a unique ray $\overrightarrow{PQ}$ such that $\angle APQ$ is a right angle. Therefore, by definition, $\overleftrightarrow{PQ} \perp \overleftrightarrow{PA}$. See Figure 2.21 above.

# Theorem 2.7.10

**Theorem 2.7.10.** At each point of a given line there is one and only one line which is perpendicular to the given line and lies in a given plane containing the line.

**Proof.** Let $P$ be a point on line $\ell$, let $\pi$ be a plane containing $\ell$, and let $A$ be a point on $\ell$ distinct from $P$. By the Plane-Separation Postulate (Postulate 12), line $\ell$ determines two halfplanes of $\pi$. Let $H$ be one of these halfplanes. By the Angle-Construction Theorem (Theorem 2.6.3), there is a unique ray $\overrightarrow{PQ}$ such that $\angle APQ$ is a right angle.

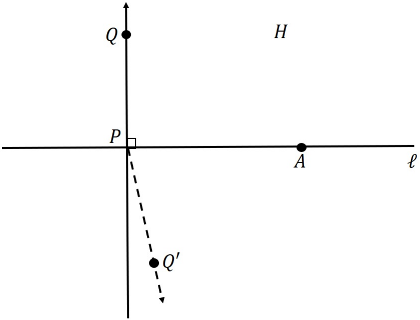



Therefore, by definition, $\overleftrightarrow{PQ} \perp \overleftrightarrow{PA}$. See Figure 2.21 above.

# Theorem 2.7.10

**Theorem 2.7.10.** At each point of a given line there is one and only one line which is perpendicular to the given line and lies in a given plane containing the line.

**Proof.** Let $P$ be a point on line $\ell$, let $\pi$ be a plane containing $\ell$, and let $A$ be a point on $\ell$ distinct from $P$. By the Plane-Separation Postulate (Postulate 12), line $\ell$ determines two halfplanes of $\pi$. Let $H$ be one of these halfplanes. By the Angle-Construction Theorem (Theorem 2.6.3), there is a unique ray $\overrightarrow{PQ}$ such that $\angle APQ$ is a right angle.




Therefore, by definition, $\overleftrightarrow{PQ} \perp \overleftrightarrow{PA}$. See Figure 2.21 above.

# Theorem 2.7.10 (continued)

**Theorem 2.7.10.** At each point of a given line there is one and only one line which is perpendicular to the given line and lies in a given plane containing the line.

**Proof (continued).** For uniqueness, we need to consider the other halfplane $H'$ determined by $\ell$. There is a unique ray $\overrightarrow{PQ'}$ such that $\angle APQ'$ is a right angle. Hence, by definition, $\overleftrightarrow{PQ'} \perp \overleftrightarrow{PA}$. To complete the proof, we must show that $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PQ'}$ are the same line. We do so by showing that $Q$, $P$, and $Q'$ are collinear. Since $\angle QPA$ and $\angle APQ'$ are supplementary adjacent angles, then by Theorem 2.7.6 rays $\overrightarrow{PQ}$ and $\overrightarrow{PQ'}$ form a linear pair, and hence $Q$, $P$, and $Q'$ are collinear. That is, the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PQ'}$ are the same and we therefore have uniqueness. □