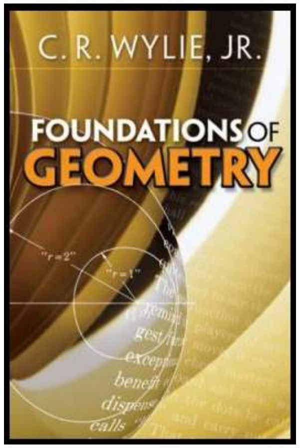
Foundations of Geometry

Chapter 2. Euclidean Geometry

2.9. The Congruence Postulate—Proofs of Theorems



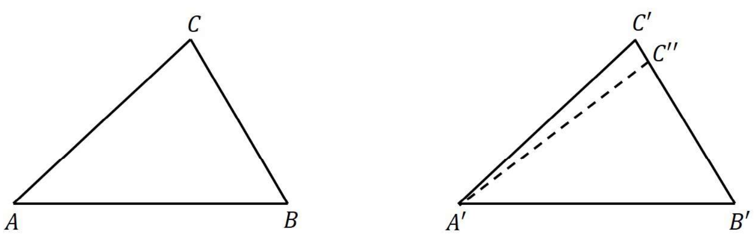
Theorem 2.9.1

Theorem 2.9.1. If there exists a one-to-one correspondence between two triangles or between a triangle and itself in which two angles and the side common to the two angles in one triangle are congruent to the corresponding parts of the other triangle, then the correspondence is a congruence and the triangles are congruent.

Proof. Let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be a correspondence between $\triangle ABC$ and $\triangle A'B'C'$ in which $\angle BAC \cong \angle B'A'C'$, $\overline{AB} \cong \overline{A'B'}$, and $\angle ABC \cong \angle A'B'C'$, as hypothesized. If $\overline{BC} \cong \overline{B'C'}$ then the triangles are congruent by Postulate 16 (The Congruence Postulate). ASSUME $\overline{BC} \ncong \overline{B'C'}$. If we get a contradiction of this assumption, then the result will follow. Suppose, without loss of generality, that $BC < B'C'$ (see Figure 2.24 below).

Theorem 2.9.1 (continued 1)

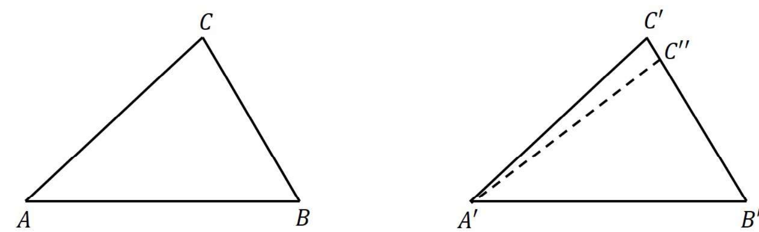
Proof (continued).



By the Point-Plotting Theorem (Theorem 2.5.5), there is a point C'' (strictly) between B' and C' such that $\overline{BC} \cong \overline{B'C''}$). Then by Postulate 16, $\triangle ABC \cong \triangle A'B'C''$. Then by the definition of congruent triangles, we have corresponding angles congruent so that $\angle B'A'C'' \cong \angle BAC$. Since $\angle BAC \cong \angle B'A'C'$ by hypothesis and congruence is transitive, then $\angle B'A'C'' \cong \angle B'A'C'$. Of course, neither C' nor C'' are not on the line $\overleftrightarrow{A'B'}$. Since B' is not between C' and C'', then C' and C'' are on the same side of $\overleftrightarrow{A'B'}$.

Theorem 2.9.1 (continued 2)

Proof (continued).



Since $\angle B'A'C'' \cong \angle B'A'C'$, then by the Angle-Construction Theorem (Theorem 2.6.3) there is a unique ray $\overrightarrow{A'X}$ such that $m_R\angle B'A'X = r$ where $r = m_R\angle B'A'C'$ (we need C' and C'' in the same halfplane determined by $\overleftrightarrow{A'B'}$ to apply the Angle-Construction Theorem). So $\overrightarrow{A'C'} = \overrightarrow{A'C''}$. Since $\overrightarrow{A'C'}$ intersects $\overline{B'C'}$ in exactly one point, then we must have $C' = C''$. But C'' is strictly between C' and B', a CONTRADICTION. So the assumption that $\overline{BC} \ncong \overline{B'C'}$ is false, and hence $\overline{BC} \cong \overline{B'C'}$, as claimed. □

Theorem 2.9.2

Theorem 2.9.2. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then $\triangle ABC \cong \triangle ACB$ and $\angle ABC \cong \angle ACB$.

Proof. Under the correspondence $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow B$ we have $\overline{AB} \cong \overline{AC}$, $\angle BAC \cong \angle CAB$, and $\overline{AC} \cong \overline{AB}$. So by Postulate 16 (The Congruence Postulate, "SAS"), $\triangle ABC \cong \triangle ACB$. Hence, by the definition of congruent triangles, $\angle ABC \cong \angle ACB$, as claimed. □

Theorem 2.9.3

Theorem 2.9.3. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then the segment determined by A and the midpoint of $\overline{BC}$ is perpendicular to $\overline{BC}$.

Proof. Let D be the midpoint of $\overline{BC}$, so that $\overline{BD} \cong \overline{CD}$. By hypothesis, $\overline{AB} \cong \overline{AC}$, and by Theorem 2.9.2 we have $\angle ABD \cong \angle ACD$. So by Postulate 16 (The Congruence Postulate, "SAS"), $\triangle ABD \cong \triangle ACD$ and hence by the definition of congruent triangles, $\angle BDA \cong \angle CDA$. Since these are congruent angles which form a linear pair, then each must have measure 90 and so are both right angles. Therefore, $\overline{AD} \perp \overline{BC}$, as claimed. □

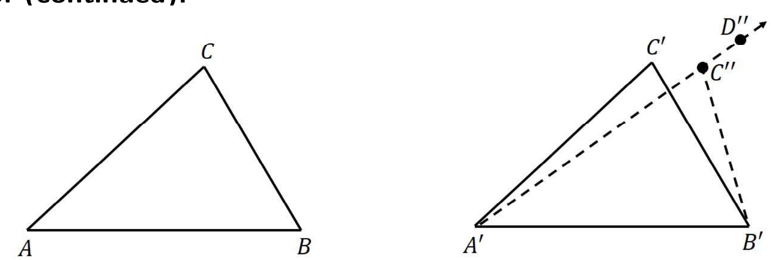
Theorem 2.9.4

Theorem 2.9.4. If there exists a one-to-one correspondence between two triangles, or between a triangle and itself, in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent.

Proof. Let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be the correspondence between $\triangle ABC$ and $\triangle A'B'C'$ for which $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$, as hypothesized. If $\angle BAC \cong \angle B'A'C'$ then the triangles are congruent by Postulate 16 (The Congruence Postulate, "SAS"). ASSUME $\angle BAC \ncong \angle B'A'C'$. If we get a contradiction of this assumption, then the result will follow. Suppose, without loss of generality, that $m\angle BAC < m\angle B'A'C'$ (see Figure 2.25 below).

Theorem 2.9.4 (continued 1)

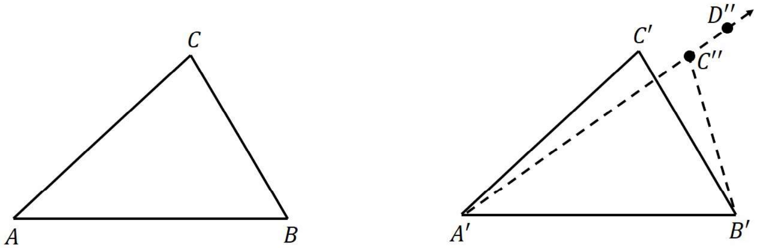
Proof (continued).



Since $m\angle BAC < m\angle B'A'C'$, then by Theorem 2.6.5 there is a unique ray $\overrightarrow{A'D'}$ between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ such that $\angle B'A'D' \cong \angle BAC$. Now by the Point-Plotting Theorem (Theorem 2.5.5) there is a point C'' on $\overrightarrow{A'D'}$ such that $\overline{A'C''} = \overline{AC}$. Since $\overrightarrow{A'D'}$ is between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$, then $\overrightarrow{A'D'}$ and C' are on the same side of $\overleftrightarrow{A'B'}$. Points C' and C'' are distinct points since $\overrightarrow{A'D'} \neq \overrightarrow{A'C'}$. By Postulate 16 (The Congruence Postulate; "ASA"), $\triangle BAC \cong \triangle B'A'C''$.

Theorem 2.9.4 (continued 2)

Proof (continued).



Since $\triangle BAC \cong \triangle B'A'C''$ then $\overline{B'C''} \cong \overline{B'C'}$, and by hypothesis $\overline{BC} \cong \overline{B'C'}$, so $\overline{B'C''} \cong \overline{BC} \cong \overline{B'C'}$. By the choice of C'' we have $\overline{C'A'} \cong \overline{C''A'}$, so $\triangle C'A'C''$ is an isosceles triangle. Since C' and C'' are on the same side of $\overleftrightarrow{A'B'}$ (as shown above), and $\overline{B'C''} \cong \overline{B'C'}$, then by the Point-Plotting Theorem (Theorem 2.5.5) we cannot have points B', C', and C'' collinear (or else rays $\overrightarrow{B'C'}$ and $\overrightarrow{B'C''}$ would coincide and we would have $C' = C''$). Since $\overline{B'C''} \cong \overline{B'C'}$ then $\triangle C'B'C''$ is an isosceles triangle.

Theorem 2.9.4 (continued 3)

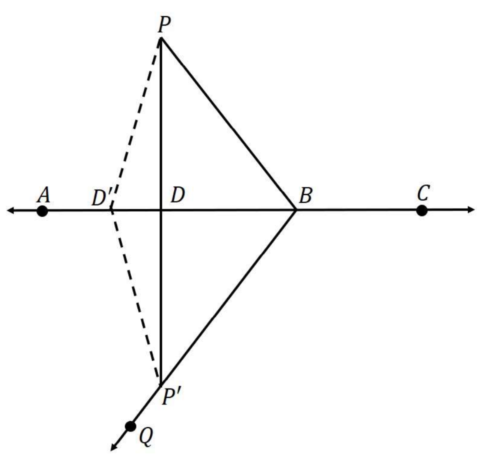
Theorem 2.9.4. If there exists a one-to-one correspondence between two triangles, or between a triangle and itself, in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent.

Proof (continued). Applying Corollary 2.9.1 to isosceles triangles $\triangle C'A'C''$ and $\triangle C'B'C''$, we have that the perpendicular bisector of $\overline{C'C''}$ passes through both point A' and B'. Since two points determine a unique line (Postulate 2), then line $\overleftrightarrow{A'B'}$ is the perpendicular bisector of $\overline{C'C''}$. But, as shown above, C' and C'' are on the same sides of $\overleftrightarrow{A'B'}$, and the halfplanes (or "sides") determined by $\overleftrightarrow{A'B'}$ are convex sets, so that $\overline{C', C''}$ must be a subset of one of the halfplanes; that is, $\overleftrightarrow{A'B'}$ cannot intersect $\overline{C'C''}$, a CONTRADICTION. So we cannot have $\angle BAC \not\cong \angle B'A'C'$, and the result now follows, as described above. □

Theorem 2.9.5

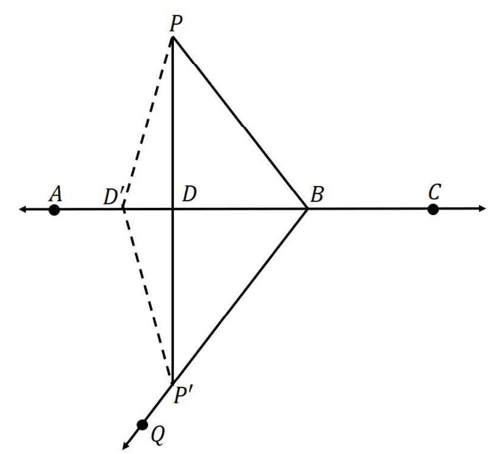
Theorem 2.9.5. Through a given point not on a given line there is one and only one line perpendicular to the given line.

Proof. Let $\overleftrightarrow{AC}$ be a line and P any point on $\overleftrightarrow{AC}$. Let B be any point between A and C. If $\overleftrightarrow{PB} \perp \overleftrightarrow{AC}$ then we have the existence of a perpendicular to the line through the given point. If $\overleftrightarrow{PB}$ is not perpendicular to $\overleftrightarrow{AC}$, then one of the angles $\angle ABP$ or $\angle PBC$ must be acute; without loss of generality, say $\angle ABP$ is acute (see Figure 2.26).



Theorem 2.9.5 (continued 1)

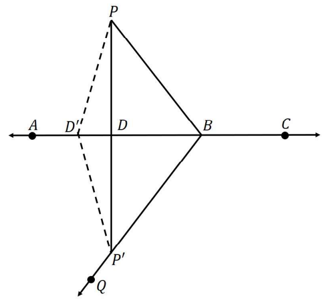
Proof (continued). By the Angle Construction Theorem (Theorem 2.6.3), there is a point Q on the opposite side of $\overleftrightarrow{AC}$ from point P such that $m\angle ABQ = m\angle ABP$. By the Point-Plotting Theorem (Theorem 2.5.5) there is a point P' on $\overleftrightarrow{BQ}$ such that $BP' = BP$. Since, by choice, points P and P' are on opposite sides of $\overleftrightarrow{AC}$, then by the Plane-Separation Postulate (Postulate 12), the segment $\overline{PP'}$ must intersect $\overleftrightarrow{AC}$, say at point D. Then by Postulate 16 (The Congruence Postulate), $\triangle PBD \cong \triangle P'BD$.



Theorem 2.9.5 (continued 2)

Proof (continued). Since $\triangle PBD \cong \triangle P'BD$, then $\angle PDB \cong \angle P'DB$. Since these angles form a linear pair, then their measure must be 90 and they are right angles. That is, $\overleftrightarrow{PP'} \perp \overleftrightarrow{AC}$ and we have shown the existence of a perpendicular to line $\overleftrightarrow{AC}$ through point P, as needed.

Now we show the uniqueness of the perpendicular. ASSUME there are is another line $\overleftrightarrow{PD'}$ that is also perpendicular to $\overleftrightarrow{AC}$. By the Point-Plotting Theorem (Theorem 2.5.5), there is a point P' on the ray $\overrightarrow{DP}$ such that $P'D = PD$. Then by Postulate 16 (The Congruence Postulate) $\triangle PDD' \cong \triangle P'DD'$. So $\angle PD'D \cong \angle P'D'D$, and $\angle P'D'D$ is also a right angle. So $\angle PD'D$ and $\angle P'D'D$ are supplementary adjacent angles and so form a linear pair. But then points P, D', and P' are collinear.



Theorem 2.9.5 (continued 3)

Theorem 2.9.5. Through a given point not on a given line there is one and only one line perpendicular to the given line.

Proof (continued). But (by Postulate 2) the distinct lines $\overleftrightarrow{PD}$ and $\overleftrightarrow{PD'}$ cannot have two distinct points (P and P' here) in common, giving a CONTRADICTION. So the assumption of the existence of a second perpendicular to $\overleftrightarrow{AC}$ through point P is false, giving uniqueness, as claimed. □