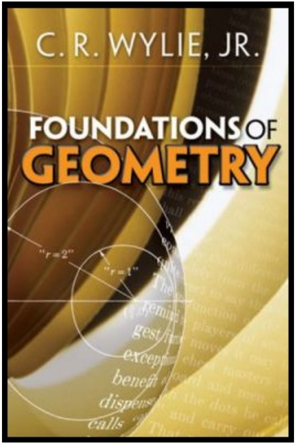
Foundations of Geometry

Chapter 2. Euclidean Geometry

2.9. The Congruence Postulate—Proofs of Theorems



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Theorem 2.9.1

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Proof. Let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be a correspondence between $\triangle ABC$ and $\triangle A'B'C'$ in which $\angle BAC \cong \angle B'A'C'$, $\overline{AB} \cong \overline{A'B'}$, and $\angle ABC \cong \angle A'B'C'$, as hypothesized. If $\overline{BC} \cong \overline{B'C'}$ then the triangles are congruent by Postulate 16 (The Congruence Postulate). ASSUME $\overline{BC} \not\cong \overline{B'C'}$. If we get a contradiction of this assumption, then the result will follow. Suppose, without loss of generality, that $BC < B'C'$ (see Figure 2.24 below).

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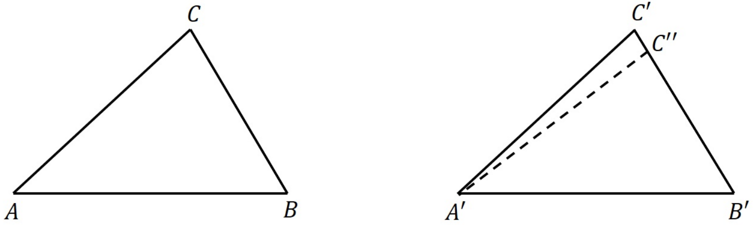
Theorem 2.9.1 (continued 1)

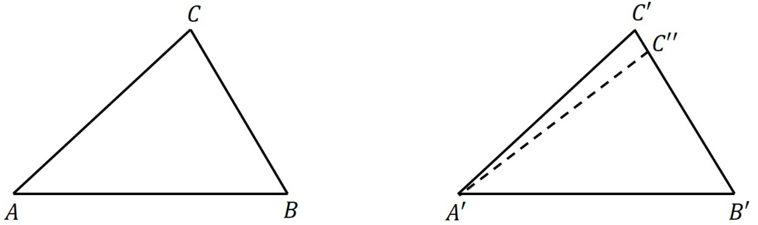
Proof (continued).



By the Point-Plotting Theorem (Theorem 2.5.5), there is a point C'' (strictly) between B' and C' such that $\overline{BC} \cong \overline{B'C''}$). Then by Postulate 16, $\triangle ABC \cong \triangle A'B'C''$. Then by the definition of congruent triangles, we have corresponding angles congruent so that $\angle B'A'C'' \cong \angle BAC$. Since $\angle BAC \cong \angle B'A'C'$ by hypothesis and congruence is transitive, then $\angle B'A'C'' \cong \angle B'A'C'$. Of course, neither C' nor C'' are not on the line $\overleftrightarrow{A'B'}$. Since B' is not between C' and C'', then C' and C'' are on the same side of $\overleftrightarrow{A'B'}$.

Theorem 2.9.1 (continued 2)

Proof (continued).



Since $\angle B'A'C'' \cong \angle B'A'C'$, then by the Angle-Construction Theorem (Theorem 2.6.3) there is a unique ray $\overrightarrow{A'X}$ such that $m_R\angle B'A'X = r$ where $r = m_R\angle B'A'C'$ (we need C' and C'' in the same halfplane determined by $\overleftrightarrow{A'B'}$ to apply the Angle-Construction Theorem). So $\overrightarrow{A'C'} = \overrightarrow{A'C''}$. Since $\overrightarrow{A'C'}$ intersects $\overline{B'C'}$ in exactly one point, then we must have $C' = C''$. But C'' is strictly between C' and B', a CONTRADICTION. So the assumption that $\overline{BC} \not\cong \overline{B'C'}$ is false, and hence $\overline{BC} \cong \overline{B'C'}$, as claimed. □

Theorem 2.9.2

Theorem 2.9.2. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then $\triangle ABC \cong \triangle ACB$ and $\angle ABC \cong \angle ACB$.

Proof. Under the correspondence $A \leftrightarrow A$, $B \leftrightarrow C$, $C \leftrightarrow B$ we have $\overline{AB} \cong \overline{AC}$, $\angle BAC \cong \angle CAB$, and $\overline{AC} \cong \overline{AB}$. So by Postulate 16 (The Congruence Postulate, "SAS"), $\triangle ABC \cong \triangle ACB$. Hence, by the definition of congruent triangles, $\angle ABC \cong \angle ACB$, as claimed. □

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Theorem 2.9.3

Theorem 2.9.3. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, then the segment determined by A and the midpoint of $\overline{BC}$ is perpendicular to $\overline{BC}$.

Proof. Let D be the midpoint of $\overline{BC}$, so that $\overline{BD} \cong \overline{CD}$. By hypothesis, $\overline{AB} \cong \overline{AC}$, and by Theorem 2.9.2 we have $\angle ABD \cong \angle ACD$. So by Postulate 16 (The Congruence Postulate, "SAS"), $\triangle ABD \cong \triangle ACD$ and hence by the definition of congruent triangles, $\angle BDA \cong \angle CDA$. Since these are congruent angles which form a linear pair, then each must have measure 90 and so are both right angles. Therefore, $\overline{AD} \perp \overline{BC}$, as claimed. $\square$

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Theorem 2.9.4

Theorem 2.9.4. If there exists a one-to-one correspondence between two triangles, or between a triangle and itself, in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent.

Proof. Let $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$ be the correspondence between $\triangle ABC$ and $\triangle A'B'C'$ for which $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{AC} \cong \overline{A'C'}$, as hypothesized. If $\angle BAC \cong \angle B'A'C'$ then the triangles are congruent by Postulate 16 (The Congruence Postulate, "SAS"). ASSUME $\angle BAC \ncong \angle B'A'C'$. If we get a contradiction of this assumption, then the result will follow. Suppose, without loss of generality, that $m\angle BAC < m\angle B'A'C'$ (see Figure 2.25 below).

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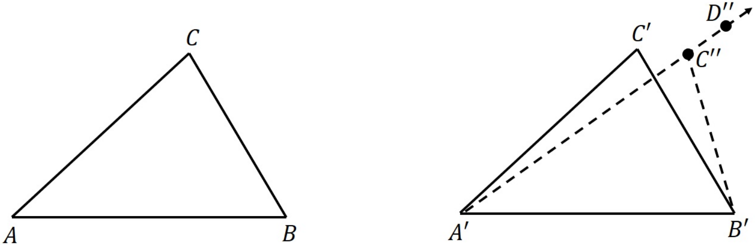
Theorem 2.9.4 (continued 1)

Proof (continued).



Since $m\angle BAC < m\angle B'A'C'$, then by Theorem 2.6.5 there is a unique ray $\overrightarrow{A'D'}$ between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ such that $\angle B'A'D' \cong \angle BAC$. Now by the Point-Plotting Theorem (Theorem 2.5.5) there is a point C'' on $\overrightarrow{A'D'}$ such that $\overline{A'C''} = \overline{AC}$. Since $\overrightarrow{A'D'}$ is between $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$, then $\overrightarrow{A'D'}$ and C' are on the same side of $\overleftrightarrow{A'B'}$. Points C' and C'' are distinct points since $\overrightarrow{A'D'} \neq \overrightarrow{A'C'}$. By Postulate 16 (The Congruence Postulate; "ASA"), $\triangle BAC \cong \triangle B'A'C''$.

Theorem 2.9.4 (continued 1)

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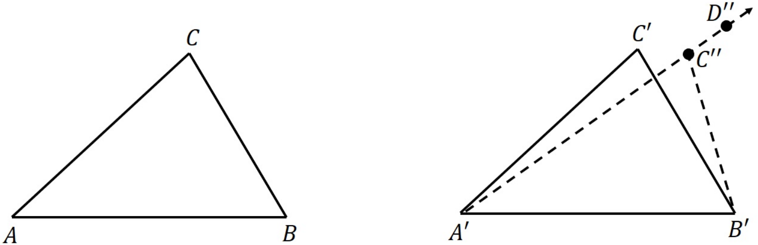

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Theorem 2.9.4 (continued 2)

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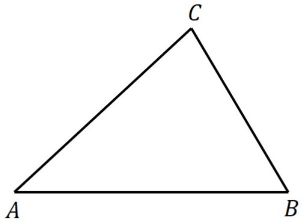

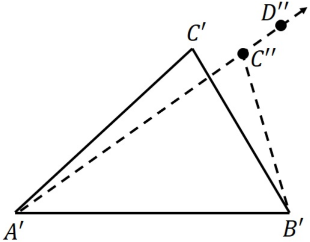

Since $\triangle BAC \cong \triangle B'A'C''$ then $\overline{B'C''} \cong \overline{B'C'}$, and by hypothesis $\overline{BC} \cong \overline{B'C'}$, so $\overline{B'C''} \cong \overline{BC} \cong \overline{B'C'}$. By the choice of C'' we have $\overline{C'A'} \cong \overline{C''A'}$, so $\triangle C'A'C''$ is an isosceles triangle. Since C' and C'' are on the same side of $\overleftrightarrow{A'B'}$ (as shown above), and $\overline{B'C''} \cong \overline{B'C'}$, then by the Point-Plotting Theorem (Theorem 2.5.5) we cannot have points B', C', and C'' collinear (or else rays $\overrightarrow{B'C'}$ and $\overrightarrow{B'C''}$ would coincide and we would have $C' = C''$). Since $\overline{B'C''} \cong \overline{B'C'}$ then $\triangle C'B'C''$ is an isosceles triangle.

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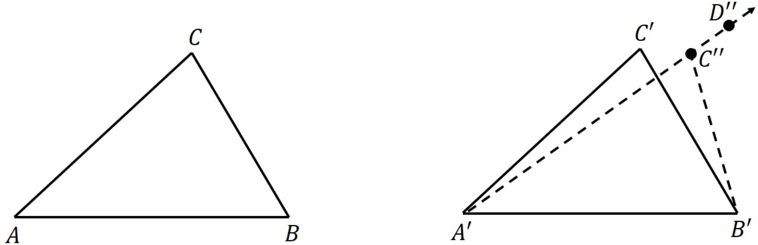

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Theorem 2.9.4 (continued 3)

Theorem 2.9.4. If there exists a one-to-one correspondence between two triangles, or between a triangle and itself, in which three sides of one triangle are congruent to the corresponding sides of the other triangle, the correspondence is a congruence and the triangles are congruent.

Proof (continued). Applying Corollary 2.9.1 to isosceles triangles $\triangle C'A'C''$ and $\triangle C'B'C''$, we have that the perpendicular bisector of $\overline{C'C''}$ passes through both point A' and B'. Since two points determine a unique line (Postulate 2), then line $\overleftrightarrow{A'B'}$ is the perpendicular bisector of $\overline{C'C''}$. But, as shown above, C' and C'' are on the same sides of $\overleftrightarrow{A'B'}$, and the halfplanes (or "sides") determined by $\overleftrightarrow{A'B'}$ are convex sets, so that $\overline{C', C''}$ must be a subset of one of the halfplanes; that is, $\overleftrightarrow{A'B'}$ cannot intersect $\overline{C'C''}$, a CONTRADICTION. So we cannot have $\angle BAC \not\cong \angle B'A'C'$, and the result now follows, as described above. □

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Theorem 2.9.5

Theorem 2.9.5. Through a given point not on a given line there is one and only one line perpendicular to the given line.

Proof. Let $\overleftrightarrow{AC}$ be a line and P any point on $\overleftrightarrow{AC}$. Let B be any point between A and C. If $\overleftrightarrow{PB} \perp \overleftrightarrow{AC}$ then we have the existence of a perpendicular to the line through the given point. If $\overleftrightarrow{PB}$ is not perpendicular to $\overleftrightarrow{AC}$, then one of the angles $\angle ABP$ or $\angle PBC$ must be acute; without loss of generality, say $\angle ABP$ is acute (see Figure 2.26).

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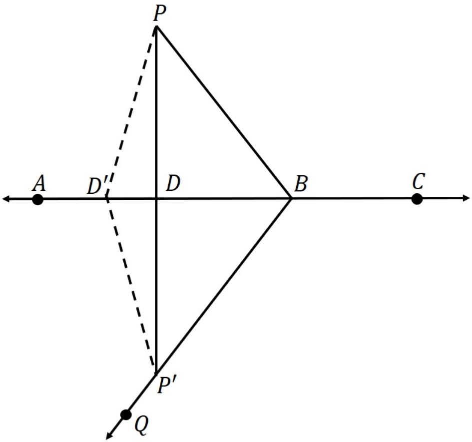

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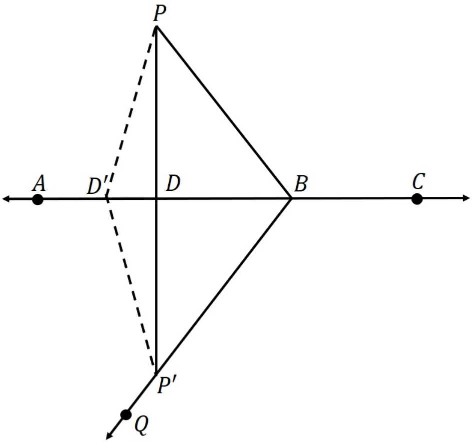
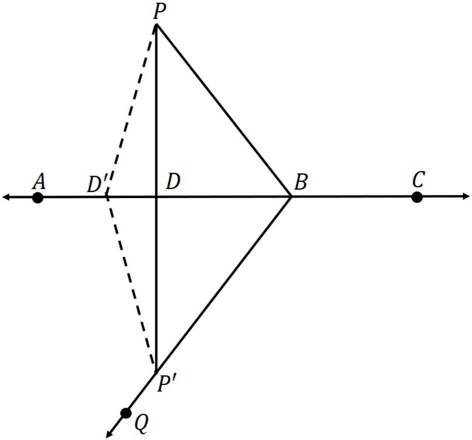


Theorem 2.9.5 (continued 1)

Proof (continued). By the Angle Construction Theorem (Theorem 2.6.3), there is a point Q on the opposite side of $\overleftrightarrow{AC}$ from point P such that $m\angle ABQ = m\angle ABP$. By the Point-Plotting Theorem (Theorem 2.5.5) there is a point P' on $\overleftrightarrow{BQ}$ such that $BP' = BP$. Since, by choice, points P and P' are on opposite sides of $\overleftrightarrow{AC}$, then by the Plane-Separation Postulate (Postulate 12), the segment $\overline{PP'}$ must intersect $\overleftrightarrow{AC}$, say at point D. Then by Postulate 16 (The Congruence Postulate), $\triangle PBD \cong \triangle P'BD$.



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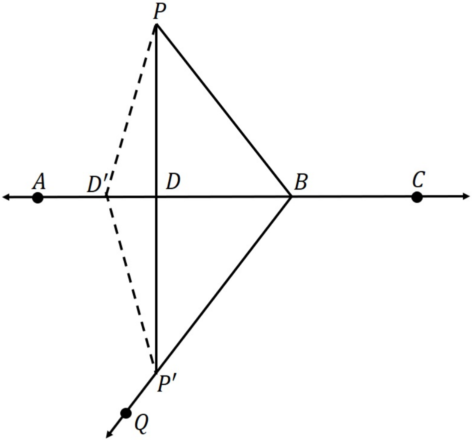

Theorem 2.9.5 (continued 2)

Proof (continued). Since $\triangle PBD \cong \triangle P'BD$, then $\angle PDB \cong \angle P'DB$. Since these angles form a linear pair, then their measure must be 90 and they are right angles. That is, $\overleftrightarrow{PP'} \perp \overleftrightarrow{AC}$ and we have shown the existence of a perpendicular to line $\overleftrightarrow{AC}$ through point P, as needed.

Now we show the uniqueness of the perpendicular. ASSUME there are is another line $\overleftrightarrow{PD'}$ that is also perpendicular to $\overleftrightarrow{AC}$. By the Point-Plotting Theorem (Theorem 2.5.5), there is a point P' on the ray $\overrightarrow{DP}$ such that $P'D = PD$. Then by Postulate 16 (The Congruence Postulate) $\triangle PDD' \cong \triangle P'DD'$. So $\angle PD'D \cong \angle P'D'D$, and $\angle P'D'D$ is also a right angle. So $\angle PD'D$ and $\angle P'D'D$ are supplementary adjacent angles and so form a linear pair. But then points P, D', and P' are collinear.

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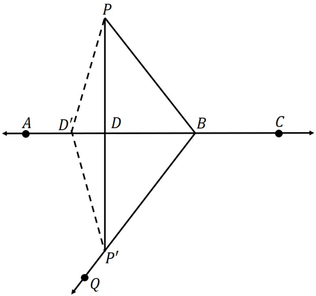

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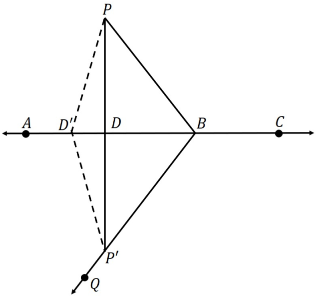

Theorem 2.9.5 (continued 3)

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Proof (continued). But (by Postulate 2) the distinct lines $\overleftrightarrow{PD}$ and $\overleftrightarrow{PD'}$ cannot have two distinct points (P and P' here) in common, giving a CONTRADICTION. So the assumption of the existence of a second perpendicular to $\overleftrightarrow{AC}$ through point P is false, giving uniqueness, as claimed. $\square$