Chapter 3. Forms

Note. This chapter gives a classification of "reflexive σ-sesquilinear forms and quadratic forms" over finite dimensional vector spaces of a finite field. Some of the results are given in terms of arbitrary fields, but our interest lies in the setting of finite fields.

3.1. σ-Sesquilinear Forms

Note.

Note. For field $\mathbb{F}$, we denote the k-dimensional vector space over $\mathbb{F}$ as $\mathrm{V}_k(\mathbb{F})$ (as introduced in Section 2.2). In these notes, we largely concentrate on finite fields. Recall that a *linear form* α is a linear map from $\mathrm{V}_k(\mathbb{F})$ to $\mathbb{F}$.

Definition. Let σ be an automorphism of field $\mathbb{F}$. A *σ-sesquilinear form* is a map b from $\mathrm{V}_k(\mathbb{F}) \times \mathrm{V}_k(\mathbb{F})$ to $\mathbb{F}$ such that $b(u, v)$ is a linear form for any fixed $v \in \mathrm{V}_k(\mathbb{F})$, the map $b(u, v)$ is additive for any fixed $u \in \mathrm{V}_k(\mathbb{F})$ (that is, $b(u, v_1 + v_2) = b(u, v_1) + b(u, v_2)$), and $b(u, \lambda v) = \lambda^\sigma b(u, v)$ (Ball is using the notation λ^σ to represent $\sigma(\lambda)$), for all $v \in \mathrm{V}_k(\mathbb{F})$ and $\lambda \in \mathbb{F}$. If σ is the identity automorphism then a σ-sesquilinear form is a *bilinear form*. Two σ-sesquilinear forms b and b' are *isometric* (or *equivalent*) if there is an isomorphism α of $\mathrm{V}_k(\mathbb{F})$ such that

$b(u, v) = b'(\alpha(u), \alpha(v))$ for all $u, v \in \mathrm{V}_k(\mathbb{F})$. A σ-sesquilinear form is *degenerate* if there is a non-zero vector $u \in \mathrm{V}_k(\mathbb{F})$ such that $b(u, v) = 0$ for all $v \in \mathrm{V}_k(\mathbb{F})$.

Note. Alternatively, a σ-sesquilinear form $b : \mathrm{V}_k(\mathbb{F}) \times \mathrm{V}_k(\mathbb{F}) \to \mathbb{F}$ is linear in the first position, additive in the second position, and interacts with scalar multiplication in the second position by introducing a power of the scalar as follows:

$$b(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 b(u_1, v) + \lambda_2 b(u_2, v) \text{ and}$$

$$b(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1^\sigma b(u, v_1) + \lambda_2^\sigma b(u, v_2).$$

Notice that an inner product is an example of a σ-sesquilinear form where σ is the identity automorphism on $\mathbb{F}$. The fact that we use inner products to define orthogonality, motives the following definition.

Definition. Let b be a σ-sesquilinear form. For any subset U of $\mathrm{V}_k(\mathbb{F})$, its *orthogonal subspace with respect to b* is

$$U^\perp = \{v \in \mathrm{V}_k(\mathbb{F}) \mid b(u, v) = 0, \text{ for all } u \in U\}.$$

If U contains only one vector, $U = \{x\}$, then we denote $U^\perp$ and either $\{x\}^\perp$ of $x^\perp$.

Lemma 3.1. *Let U be a subspace of $\mathrm{V}_k(\mathbb{F})$. If b is a non-degenerate σ-sesquilinear form on $\mathrm{V}_k(\mathbb{F})$ then $\dim U + \dim U^\perp = k$.*

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