Chapter 4. Geometries

Note. Simeon Ball starts the preface of the book with the statement (see page ix): "This book is essentially a text book that introduces the geometrical objects which arise in the study of vector spaces over finite fields." In this chapter we will refer to vector spaces over fields and our interest lies with the case of finite fields. So all fields $\mathbb{F}$ will be assumed to be finite.

4.1. Projective Spaces

Note. The idea of a projective space is to project from the zero vector in such a way that all vectors lying in the same "direction" (well, $\pm$) from the origin are considered equivalent. This equivalence can be dealt with in terms of an equivalence relation in which these vectors are in the same equivalence class.

Note. For field $\mathbb{F}$, we denote the k-dimensional vector space over $\mathbb{F}$ as $\mathrm{V}_k(\mathbb{F})$ (as introduced in Section 2.2).

Definition. The *projective space* $\mathrm{PG}_{k-1}(\mathbb{F})$ has as its points the one-dimensional subspaces of $V_k(\mathbb{F})$, has as its *lines* the two-dimensional subspaces of $\mathrm{V}_k(\mathbb{F})$, and in general the $(d-1)$-dimensional *subspaces* of $\mathrm{PG}_{k-1}(\mathbb{F})$ are the d-dimensional subspaces of $\mathrm{V}_k(\mathbb{F})$. A *hyperplane* in $\mathrm{PG}_{k-1}(\mathbb{F})$ is a $(k-2)$-dimensional subspace of $\mathrm{PG}_{k-1}(\mathbb{F})$ (or, equivalently, a $(k-1)$-dimensional subspace of $\mathrm{V}_k(\mathbb{F})$.

Note. With the field $\mathbb{F}_2 = \mathbb{Z}_2$, the vector space $\mathrm{V}_3(\mathbb{F}_{\not\geq})$ contains $2^3 = 8$ vectors. These 8 vectors give 7 one dimensional subspaces of $\mathrm{V}_3(\mathbb{F}_2)$ (each consisting of two vectors, the zero vector and a nonzero vector). These give 7 points of the projective space $\mathrm{PG}_2(\mathbb{F}_2)$. Any two distinct nonzero vectors of $\mathrm{V}_3(\mathbb{F}_2)$ determine a two-dimensional subspace containing the zero vector and three other vectors (the two basis vector and their sum; a vector in $\mathrm{V}_3(\mathbb{F}_2)$ is its own additive inverse). So the projective space has 7 points and $\binom{7}{2}/3 = 7$ lines (since there are $\binom{7}{2}$ ways to choose the two nonzero vectors and each line contains three pairs of nonzero vectors). The projective space $\mathrm{PG}_2(\mathbb{F}_2)$ is called the *Fano plane* and is given in Figure 4.1.

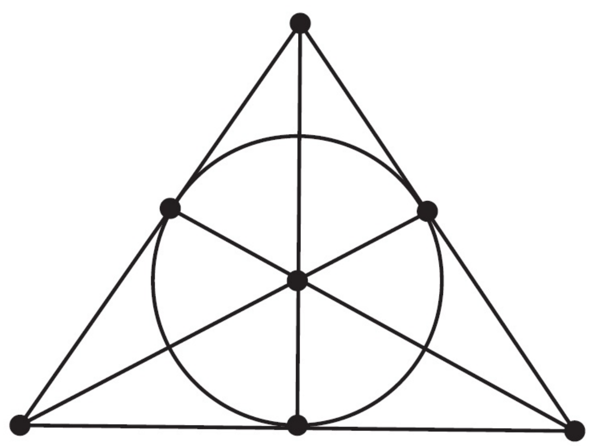

Figure 4.1 The Fano plane $\mathrm{PG}_2(\mathbb{F}_2)$.

Definition. Let P be a set whose elements are points. Let M be a set of subsets of P that include all the singleton subsets of P. The (P, M) is a *set system*. Two set systems (P, M) and (P', M') are *isomorphic* if there is a bijection from P to P' which induces a bijection from M to M'.

Note. A projective space $\mathrm{PG}_{k-1}(\mathbb{F})$ is a set system where the points determine the singletons and the subspaces of $\mathrm{PG}_{k-1}(\mathbb{F})$ determine the other subsets of the set system. Let $\mathrm{PG}_{k-1}(\mathbb{F})^*$ denote the projective space whose points are the hyperplanes of $\mathrm{V}_k(\mathbb{F})$ and where, for each nontrivial r-dimensional subspace U of $\mathrm{V}_k(\mathbb{F})$, we have a subspace of $\mathrm{PG}_{k-1}(\mathbb{F})^*$ consisting of the hyperplanes containing U. We are now ready to prove our first result of this chapter.

Theorem 4.1. The set system $\mathrm{PG}_{k-1}(\mathbb{F})$ is isomorphic to $\mathrm{PG}_{k-1}(\mathbb{F})^*$.

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