Section 1.3. Properties of Angles.

Note. In this section, we consider angles in a triangle and angles determined by an arc of a circle. We give very informal "proofs" of the results in this section. This includes material from Books I and III of Euclid's *Elements*.

Note. In Figure 1.7 (left and middle) we see parallel lines and some angles which are equal. The angles in Figure 1.7(left) are often called *alternate interior angles*, and the angles in Figure 1.7 (middle) are often called *corresponding angles*. The third line/line segment/ray (other than the parallel lines) in Figure 1.7 (left and middle) is called a *transversal*. In Figure 1.7 (right), we have that the two unlabeled angles in the two triangles are equal since they are *opposite angles* (or "vertically opposite angles"). Since the sums of the angles of a triangle are π radians or $180°$ (as to be argued below), then the "orthogonal angles" α and β must be equal. However, for now, we take these claims as true and don't really bother with proofs. We will explore these ideas more in Chapter 2, "The Elements of Euclid" (see Section 2.1. Book I).

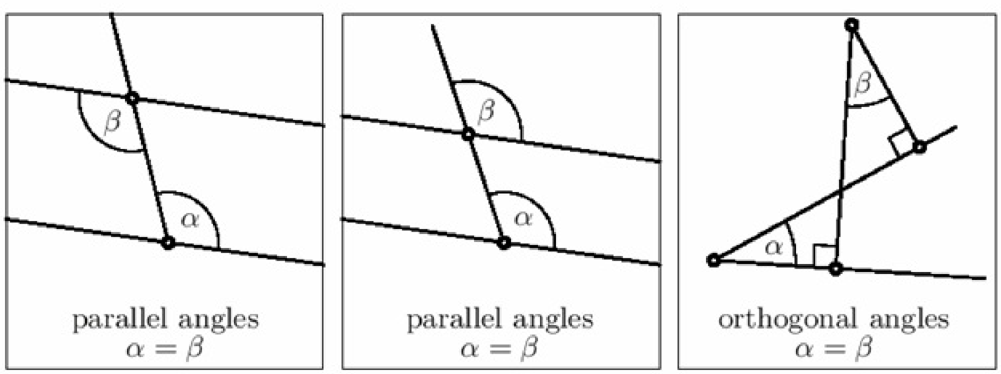

Fig. 1.7. Parallel and orthogonal angles

Note. Figure 1.8 is used to argue the following:

Theorem 1.2. (Euclid, I.32). The sum of three angles of an arbitrary triangle ABC is equal to two right angles: $\alpha + \beta + \gamma = 180°$.

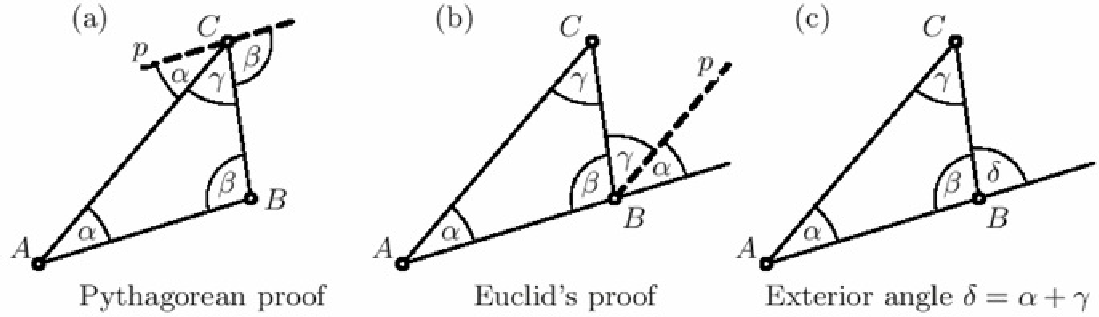

Fig. 1.8. Angles in a triangle (Eucl. I.32)

The Pythagorean proof of Theorem 1.2 is based on introducing a line through point C which is parallel to side AB of the triangle. Then we conclude that the introduced angles α and β are the sizes claimed since the angles labeled with α's are alternate interior angles (with transversal AC) and the angles labeled with β's are alternate interior angles (with transversal BC). Now $\alpha + \beta + \gamma = 180°$ (since they form a "straight angle"), as claimed.

Euclid's proof extends side AB and introduces line p parallel to side AC. Then we conclude that the introduced angles α and γ are the sizes claimed since the angles labeled with α's are corresponding angles (with transversal extended side AB) and the angles labeled with γ's are alternate interior angles (with transversal BC). Now $\alpha + \beta + \gamma = 180°$ (since they form a "straight angle"), as claimed.

Note. In Figure 1.8(c), the rightmost angles α and γ in Figure 1.8(b) are combined to form exterior angle δ. Based on Euclid's proof, we now have:

Corollary 1.3. Each exterior angle is the sum of the non-adjacent interior angles: $\delta = \alpha + \gamma$ (see Figure 1.8(c)).

Note. In Figure 1.9(a), we have a circle with center O and diameter AB. We choose an arbitrary point C (where $A \neq C \neq B$) and join it to A and to O. Now triangle AOC is isosceles since two of its sides are radii of the circle. So (as we will argue in Euclid I.5 in Section 2.1. Book I), we have the same angle β at A as at C. Now angle BOC is an exterior angle to triangle AOC, so by Corollary 1.3 angle BOC is twice angle BAC. We call angle BOC the *central angle on arc* BC, and angle BAC is an *inscribed angle on arc* BC. Next we choose an arbitrary point D on the circle such that C and D are on opposite sides of the diameter AB; see Figure 1.9(b). Then deleting diameter AB, we get Figure 1.9(c). We label the size of the inscribed angle on arc CD (angle CAD) as α and we now know that the size of the central angle on arc CD (angle COD) is 2β. We started with arbitrary point C and only required that points C and D lie on opposite sides of a diameter (some such a diameter always exists). Notice that no restriction has been placed on the angles of β and γ (though each must be less than 90°), so we have the following result (see also Figure 1.9(d)).

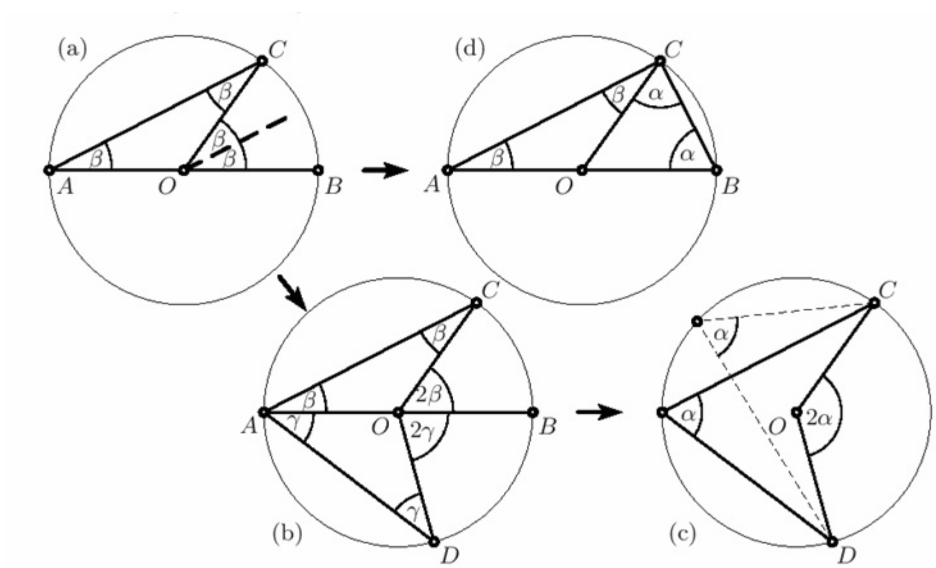

Fig. 1.9. Central angle and inscribed angle

Theorem 1.4. (Euclid, III.20). A central angle of a circle is twice any inscribed angle on the same arc.

Note. We can now easily show the following, either using Theorem 1.2 or Theorem 1.4.

Theorem 1.5. (Euclid, III.31). If AB is the diameter and C is a point (other than A or B) on the circle, then angle ACB is a right angle.

"Proof." We refer to Figure 1.9(d). Applying Theorem 1.2 to triangle ABC we have that $2\alpha + 2\beta = 180^\circ$, and so $\alpha + \beta = 90^\circ$, as claimed. ∎

Note. The converse of Theorem 1.5 also holds, as is to be shown in Exercise 2.4.

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