Section 1.4. The Regular Pentagon.

Note. In this section, we explore the length of a diagonal of a pentagon and show that it is irrational. This leads to a brief discussion of the golden ratio and irrational numbers.

Note 1.4.A. By introducing the diagonals of a regular pentagon, we get the star shown in Figure 1.10(b) (where the regular pentagon is inscribed in a circle). We let Φ denote the length of the diagonal of a regular pentagon with each side of length 1. The central angles associated with arcs AB, BC, CD, DE, and AE are each of size $360°/5 = 72°$. The inscribed angles of each of these arcs is $\alpha = 72°/2 = 36°$ by Theorem 1.4 (these are the angles in the star). Consider a triangle ACD determined by two diagonals of the star and an included edge of the pentagon (see Figure 1.10(c)). It contains a smaller triangle CDF, which we claim is similar to triangle ACD. To justify this claim, we observe that the central angle is in an isosceles triangle so that the other two angles must be $(180 - 72)/2° = 54° = 3\alpha/2$, and the interior angle of the regular pentagon is $2(3\alpha/2) = 3\alpha = 108°$. See Figure 1.10.A.

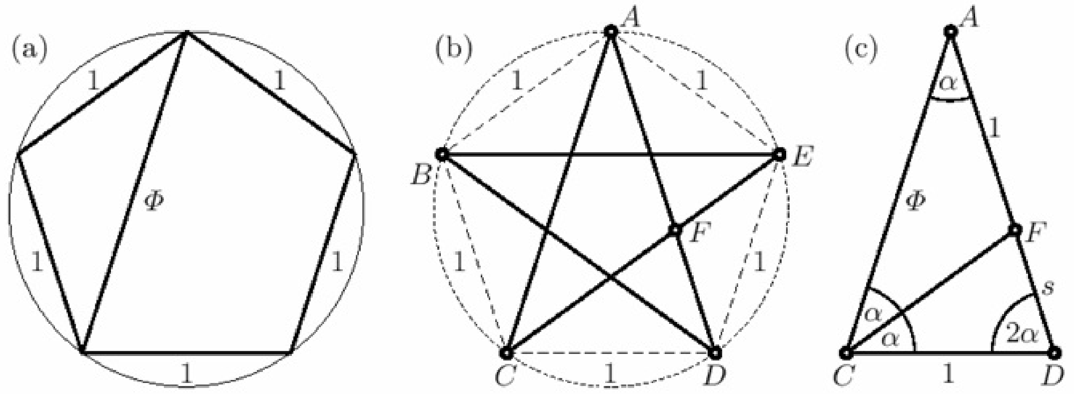

Fig. 1.10. The regular pentagon

So the angles have sizes as given in Figure 1.10(c). Notice that triangles ACD and CDF have two equal angles (and hence all three angles are equal by Theorem 1.2). That is, the triangles are similar. So $\frac{s}{1} = \frac{1}{\Phi}$. Since triangle ACD is isosceles, then $\Phi = 1 + s$. Eliminating s gives the equation $\Phi = 1 + 1/\Phi$ or $\Phi^2 = \Phi + 1$. Eliminating Φ gives the equation $1/s = 1 + s$ or $1 = s + s^2$. It is to be shown geometrically in Exercise 2.15 that $\Phi = \frac{\sqrt{5}}{2} + \frac{1}{2}$ and $s = \frac{\sqrt{5}}{2} - \frac{1}{2}$. (Of course we can algebraically use the quadratic equation to show these as well.) The number Φ is called the *golden ratio*.

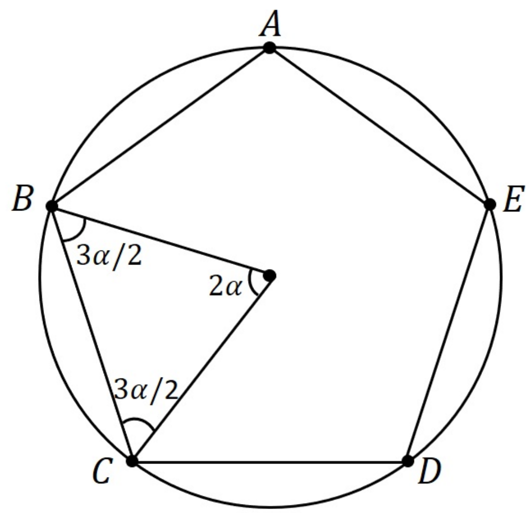

Figure 1.10.A.

Note 1.4.B. Wanner and Ostermann state that "...many beautiful ancient buildings, in particular the *Parthenon* on the Acropolis, fit so perfectly into a 'golden rectangle' (a rectangle with sides of length 1 and Φ)..." (see page 10). This is a popular story among amateur mathematicians. However, it is very questionable. In Mario Livio's *The Golden Ratio — The Story of Phi, the World's Most Astonishing Number*, New York: Broadway Books (2002) it is commented:

> "The appearance of the Golden Ratio in the Parthenon was seriously questioned by University of Maine mathematician George Markowsky in his 1992 *College Mathematics Journal* article 'Misconceptions about the Golden Ratio' ... Using the numbers quoted by Marvin Trachtenberg and Isabelle Hyman in their book *Architecture: From Prehistory to Post-Modernism* (1985), I am not convinced that the Parthenon has anything to do with the Golden Ratio." (page 74)

More details can be found online in Mario Livio's "The golden ratio and aesthetics", *Plus Magazine*, November 1, 2002 (accessed 8/7/2021).

Note 1.4.C. We now show that Φ is irrational. Suppose to the contrary that $\Phi = m/n$. Since $\Phi = 1 + 1/\Phi$, as shown above, then we must have

$$\frac{m}{n} = 1 + \frac{1}{m/n} = 1 + \frac{m}{n} = \frac{m+n}{m}.$$

But if m and n are relatively prime, then so are m and $m+n$ (as shown in Euclid VII.2; we'll explore this in Section 2.4). So if m and n are relatively prime then m/n is in unique reduced form of Φ. Since m and $m+n$ are relatively prime then $(m+n)/n$ is also the unique reduced form of Φ. But then we cannot have $m/n = (m+n)/m$ since $m \neq n$ ($\Phi \neq 1$, for example). This contradiction shows that Φ is not rational.

Note 1.4.D. Proclus (412 AD – 485 AD) in his commentaries on Euclid credits Pythagoras with the discovery of the study of irrational numbers (commonly called *incommensurables* by the Greeks). Wanner and Ostermann mention: "The fact that the regular pentagon, considered holy by the Pythagoreans, has a non-measurable [i.e., irrational] diagonal was a real shock. A legend says that Hippasus, having discovered this fact and talked too much, was drowned at sea." See page 11. Iamblichus (245 AD – 325 AD) ties the story to Hippasus, as described in Julian Havil's *The Irrationals: A Story of the Numbers You Can't Count On*, Princeton University Press, (2012):

> "...tradition has it that the author of the destruction of the Pythagorean ideal was none other than an acolyte: Hippasus of Metapontum, a man who has a decidedly negative Pythagorean press. Not only is he accused of destroying the concept of commensurability, he is meant to have spoken of the horror outside the secretive Pythagorean community—and he is meant to have done the same with his discovery that a dodecahedron can be inscribed in a sphere."

It is in Iamblichus' *On the Pythagorean Way of Life* that the drowning of Hippasus is mentioned.

Hippasus engraving by Girolamo Olgiati (1580) from the

Wikipedia page on Hippasus