Section 1.5. The Computation of Areas.

Note. In this section, we introduce "the third pillar of this chapter," as Wanner and Osermann call it (see page 11), along with Thales' Theorem (Theorem 1.1) and Euclid III.20 (Theorem 1.4).

Note/Definition. We start with the idea or definition that the *area of a rectangle* with sides of lengths a and b is $\mathcal{A} = a \cdot b = ab$.

Note. Based on the area of a rectangle, we claim that the area of a parallelogram of base length a and height h is $\mathcal{A} = a \cdot h = ah$. This is Euclid's I.35 which is justified by Figure 1.11 (second box) by cutting off a right triangle from one side of the parallelogram and moving it to the other side to produce a rectangle of base a and height h (and hence a rectangle of area ah, the same area as that of the given parallelogram).

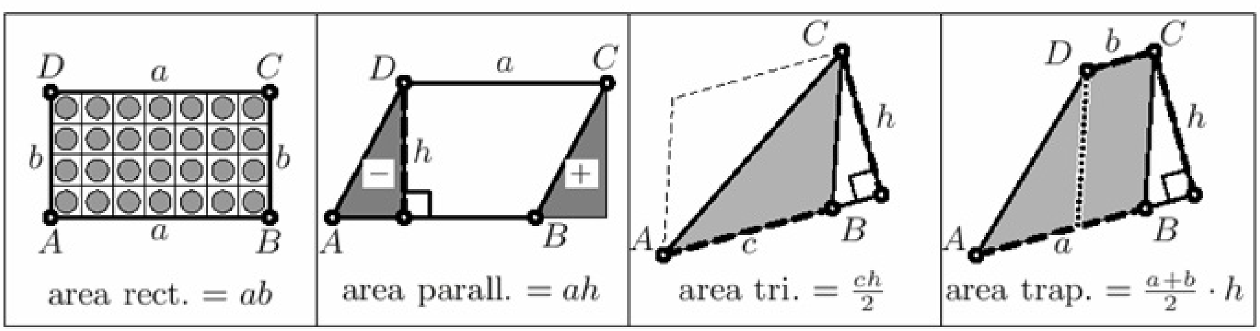


Fig. 1.11. Areas of rectangle, parallelogram, triangle and trapezium

Note/Definition. The *area of a triangle* of base c and height a is half the area of the parallelogram with base c and height h (namely, half of $c \cdot h = ch$):

$$\mathcal{A} = \textit{ area of triangle } = \text{base } \times \text{ altitude divided by 2 } = \frac{c \cdot h}{2}.$$

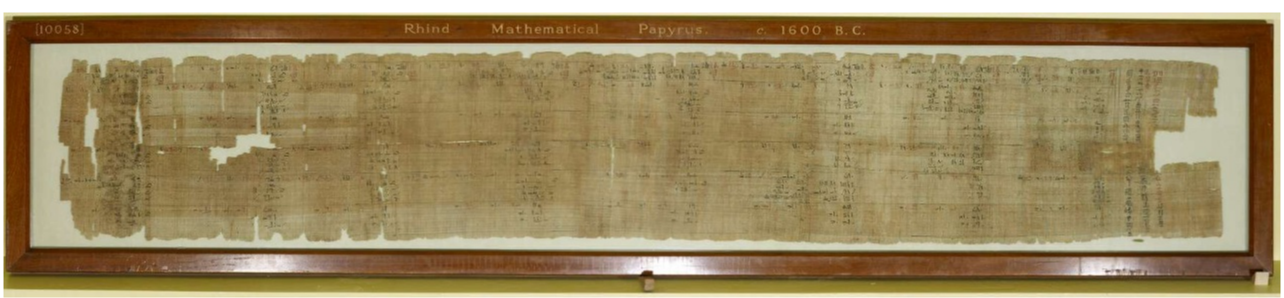
See Figure 1.11 (third box) for motivation/justification.

Note/Definition. The *area of a trapezium* with base lengths a and b and with height h is

$$\mathcal{A} = bh + \frac{a-b}{2} \cdot h = \frac{2bh + (a-b)h}{2} == \frac{ah + bh}{2} = \frac{a+b}{2}h.$$

This is motivated/justified in by cutting the trapezium into a parallelogram and a triangle and using the two previous notes/definitions.

Note. The Rhind papyrus (also called the "Ahmes papyrus" after the scribe who wrote it) was bought by egyptologist A.H. Rhind in 1858 in a market in Luxor, Egypt. It was written around 1650 BCE and is claimed to be a copy of an original work from the 19th centurey BCE.



An image of the Rhind papyrus from the The British Museum

It is described on the The British Museum webpage (accessed 8/14/2021) as "The papyrus is probably a mathematics textbook, used by scribes to learn to solve particular mathematical problems by writing down appropriate examples. Eighty-four problems are included in the text covering tables of divisions, multiplication, and handling of fractions; and geometry, including volumes and areas."

Note. The Egyptians used a decimal system with the symbols: 1 = 𓏤, 10 = 𓎆, 100 = 𓍢, 1,000 = 𓆼, 10,000 = 𓂭, 100,000 = 𓆐, and 1,000,000 = 𓁨 (these fonts are generated in LaTeX using the package `hieroglf`). They then repeated symbols as needed to represent multiples of these numbers. For example,

(notice that the symbol for 100 here from Ostermann and Wanner is slightly different from that given in LaTeX above).

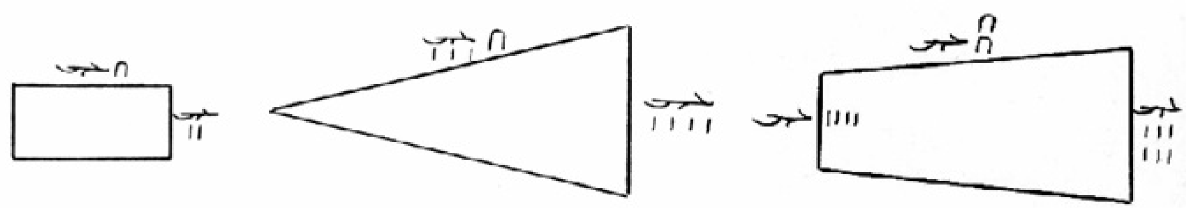
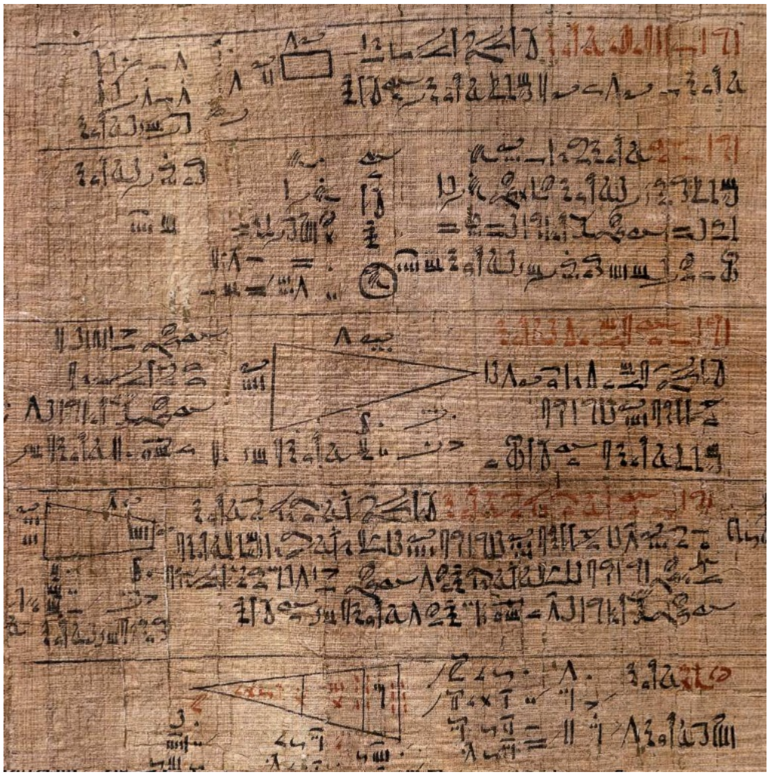


Fig. 1.12. Area calculations in Rhind papyrus; rectangle $10 \cdot 2 = 20$ (No. 49, left); triangle $\frac{4 \cdot 10}{2} = 20$ (No. 51, middle); trapezium $\frac{4+6}{2} \cdot 20 = 100$ (No. 52, right); reproductions of transcriptions by Peet (1923)

In Figure 1.12 (left) we have Problem 49 which gives a rectangle of dimensions 2 = 𓏤𓏤 by 10 = 𓎆, the area of which is 20 = 𓎆𓎆 (though there seems to be an error by the scribe at this point). In Figure 1.12 (center) we have Problem 51 which gives a triangle of base 4 = 𓏤𓏤𓏤𓏤 and height 10 = 𓎆, the area of which is 20 = 𓎆𓎆. In Figure 1.12 (right) we have Problem 52 which gives a trapezium of base lengths 4 = 𓏤𓏤𓏤𓏤 and 6 = 𓏤𓏤𓏤𓏤𓏤𓏤, and height 20 = 𓎆𓎆, the area of which is $\frac{(4)+(6)}{2}(20) = 100$ = 𓍢.



An image of the part of the Rhind papyrus including the drawings from Figure 1.12, from the The British Museum

See the MacTutor Mathematics in Egyptian Papyri website for more of the problems in the Rhind papyrus (accessed 8/14/2021).

Note. Now we consider the areas of similar triangles. We use Figure 1.2 (lower left) to give circumstantial evidence for the relationship between areas of similar triangles. Here, we consider the three triangles ABC, $AB'C'$, and $A, B''C''$. Notice that all three triangles are similar. Triangle ABC has sides 5 times as long as the sides of $AB''C''$, and $AB'C'$ has sides 8/5 as long as the sides of ABC. We use the congruent copies of the little triangle $AB''C''$ to discuss area. We use the red lines in the modified version of Figure 1.2 to count copies of the little triangle. The number of copies of the little triangle in ABC is $1+3+5+7+9 = 25 = 5^2$. The number of copies of the little triangle in $AB'C'$ is $1+3+5+7+9+11+13+15 = 64 = 8^2$. Notice that the ratio of the area of $AB'C'$ to the area of ABC is $64/52 = 8^2/5^2 = (8/5)^2$. So if two triangles are similar with rational ratio r between the lengths of their sides (such as triangles $AB'C'$ and ABC, where the ratio is $r = 8/5$) then the ratio of the areas is r^2 (such as triangles $AB'C'$ and ABC, where the ratio is $r^2 = (8/5)^2$).

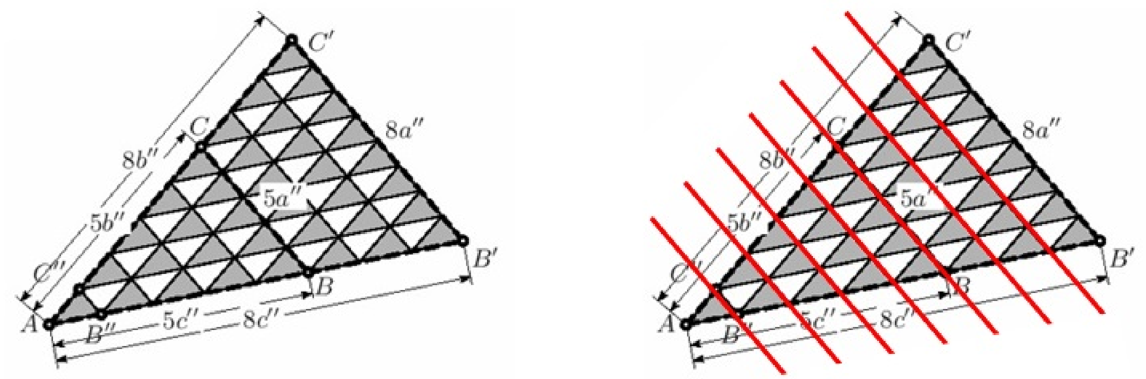
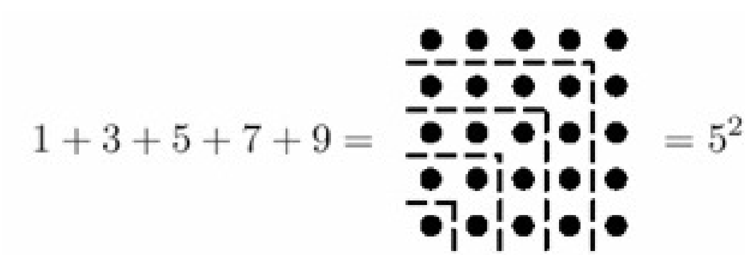


Figure 1.2 lower left and modified

We used the red lines to partition the larger triangle into collections of odd numbers of little triangles (which are sort a greatest common divisor of the larger triangles) and then counted the little triangles. The fact that we get areas expressed in terms of perfect squares (25 and 64 for triangles ABC and $AB'C'$) is illustrated in the following (from page 12 of Ostermann and Wanner).



This argument can be cleaned up and turned into a rigorous proof, provided the ratio between the lengths of the similar triangles is rational. Not surprisingly, the same result holds even if the ratio is irrational (though this line of reasoning does not apply to the irrational case), as stated next.

Theorem 1.6. (Euclid VI.19) A similar triangle with q times longer sides has q^2 times larger area.

Revised: 8/14/2021