Section 1.7. The Pythagorean Theorem

Note. We saw in the previous section that the Pythagorean Theorem was know (at least in some cases) 3,600 to 4,000 years ago.

Note. Figure 1.15 gives a visual presentation of three proofs of the Pythagorean Theorem from three "civilizations," Chinese, Indian, and Arabic. With c as the length of the hypotenuse, we consider a square of area c^2 in Figure 1.15(a).

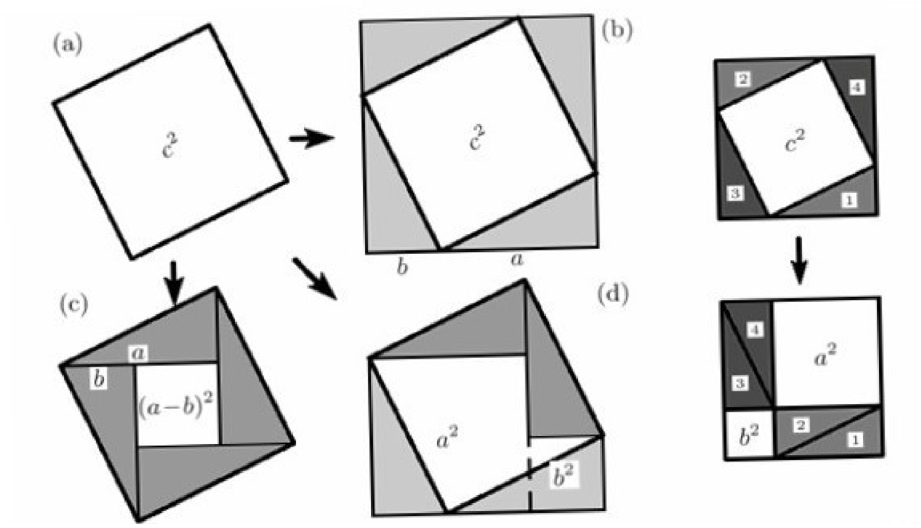


Figure 1.15

Note. In Figure 1.15(b), four right triangles with legs of lengths a and b and hypotenuse with length c are introduced. Based on the fact that the angles of a triangle sum to 180°, we can show that Figure 1.15(b) actually is a square. Now we can use the fact that the area of a triangle is 1/2(base)(altitute) (see Section 1.5. The Computation of Areas) to show that the area of the square of Figure 1.15(b) is $(a+b)^2 = 4 \times \left(\frac{1}{2}ab\right) + c^2$ or $a^2+2ab+b^2 = 2ab+c^2$ or $a^2+b^2 = c^2$. The text book credits this proof to Chou-pei Suan-ching of China in 250 BCE (the reference on this is B.L. van der Waerden, *Geometry and Algebra in Ancient Civilization*, Berlin: Springer-Verlag (1983)). The triangles can also be sifted around to represent the same square of Figure 1.15(b) in terms of two square (of areas a^2 and b^2) and two a by b rectangles (see Figure 1.15, right).

Note. In Figure 1.15(c) four right triangles are *removed* from the square of area c^2. Again, since the sum of the angles of a triangle is 180° then the triangles "fit" together to form the configuration of Figure 1.15(c). Computing areas gives $c^2 = 4 \times \left(\frac{1}{2}ab\right) + (a-b)^2$ or $c^2 = 2ab + (a^2 - 2ab + b^2)$ or $a^2 + b^2 = c^2$. The text book attributes this proof to the Indian Bhāskara II (1114–1185), but does not give a reference. A solid reference about this is Kim Plofker's "Mathematics in India," in Victor Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Princeton University Press (2007) (see pages 476–477).

Note. Yet another, but similar, proof is illustrated in Figure 1.15(d). This is a combination of the two methods above, in that two right triangles with legs of lengths a and b are added to the outside of the square of area c^2 and two such right triangles are "subtracted" from inside the square. So the result (in white and light gray in Figure 1.15(d)) is the same as the area of the original square (namely, c^2) and equals $a^2 + b^2$ as can be seen in the figure (which is, again, justified by the fact that the angles of a triangle sum to 180°). This proof is attributed to Thâbit ibn Qurra (828–901); see F. J. Swetz's *From Five Fingers to Infinity*, Open Court (1996). A webpage with animations illustrating these three proofs, and many others, is available on the Many Proofs of Pythagorean Theorem webpage.

Note. Consider the figure below (the left part). We first give an informal argument for the Pythagorean Theorem. To each side of right triangle $AB\Gamma$ has been attached a square. We want to show that the area of square $B\Gamma E\Delta$ equals the sum of the areas of squares $ABZH$ and $A\Gamma K\Theta$. The areas of the two dark grey triangle $BA\Delta$ and $BZ\Gamma$ are the same, since one triangle can be obtained from the other by rotating through $90°$ about point B. The triangle $BZ\Gamma$ has the same base and altitude as the square $BAHZ$ (the common base is the length of the segment BZ and the common height is the length of segment AB). The triangle $BA\Delta$ has the same base and height as the rectangle $B\Delta\Gamma P$ (the common base is the length of line segment $B\Delta$ and the common height is the length of segment $\Delta\Gamma$). So half the area of square $ABZH$ equals half the area of rectangle $B\Delta\Lambda P$, and hence the area of $ABZH$ equals the area of rectangle $B\Delta\Lambda P$. Similarly, the area of square $A\Gamma K\Theta$ equals the area of rectangle $\Lambda E\Gamma P$. Since square $B\Gamma E\Delta$ is composed of the two rectangles $B\Delta\Lambda P$ and $\Lambda E\Gamma P$, then the sum of the areas of squares $ABZH$ and $A\Gamma K\Theta$ equals the area of square $B\Gamma E\Delta$, as needed.

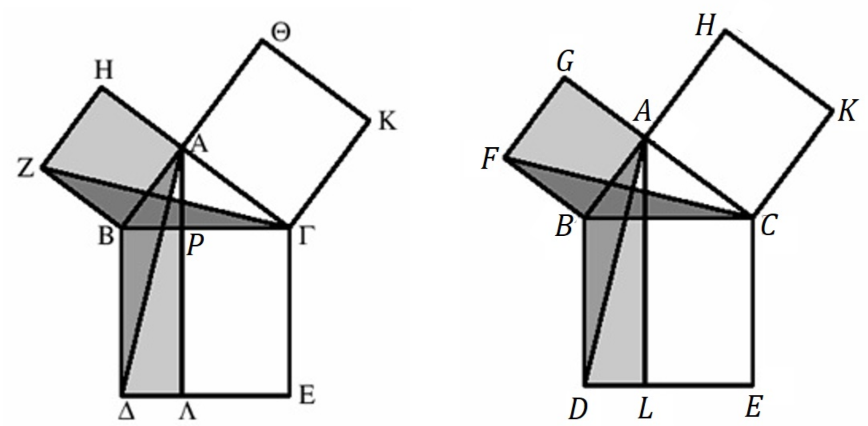


Figure. Part of Figure 1.19 (left, with the label P added) and a modification of it similar to the figure for the Pythagorean Theorem given in Euclid's *Elements*.

Note. The argument above is basically the argument made by Euclid in his *Elements.* This is such a historical result given in a historical reference, we now reproduce Euclid's proof as stated in Thomas Heath's *The Thirteen Books of Euclid's Elements, translated from the test of Heiberg, with introduction and commentary*, Second Edition, Cambridge: Cambridge University Press (1926) (reprinted in 1956 by Dover Publications). A copy is online at bibotu.com (accessed 9/10/2021). The bold-faced items are references to other results in the *Elements.*

Proposition I.47. *In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.*

Proof. Let ABC be a right-angled triangle having the angle BAC right;

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC; [**I.46**]

through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

Then since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

therefore CA is in a straight line with AG. [**I.14**]

For the same reason BA is also in a straight line with AH.

And, since the angel DBC is equal to the angle FBA: for each is right: let the angle ABC be added to each;

therefore the whole angle DBA is equal to the whole angle FBC. [**C.N.2**]

And, since DB is equal to BC, and FB to BA, the two sides AB, BD are equal to the two sides FB, BC respectively, and the angle ABD is equal to the angle FBC

therefore the base AD is equal to the base FC, and the triangle ABD is equal to the triangle FBC. [**I.4**]

Now the parallelogram BL is double of the triangle ABD, for they have the same base BD and are in the same parallels BD, AL. [**I.41**]

And the square GB is double of the triangle FBC, for they again have the same base FB and are in the same parallels FB, GC. [**I.41**]

[But the doubles of equals are equal to one another.]

Therefore the parallelogram BL is also equal to the square GB.

Similarly, if AE, BK be joined, the parallelogram CL can also be proved equal to the square HC;

therefore the whole square $BDEC$ is equal to the two squares GB, HC. [**C.N.2**]

And the square $BDEC$ is described on BC, and the squares GB, HC on BA, AC.

Therefore the square on the side BC is equal to the squares on the sides BA, AC.

Therefore, etc. Q. E. D.

Note. Vatican Manuscript Number 190 dates from the 10th century and contains the Books I to XII of *Elements*, along with some other work. The text displays properties indicating that it is a more ancient version than others that survive. The image here (from the Greek Mathematics and its Modern Heirs webpage) is from the page containing the proof of the Pythagorean Theorem.

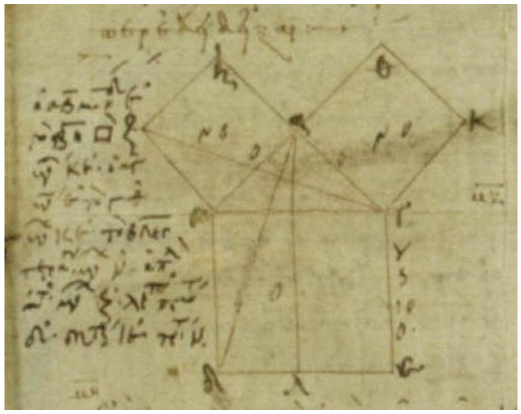


Figure. Vatican Manuscript Number 190, volume 1 folio 39 (close-up).

Note. Consider the right triangle ABC in Figure 1.20. Introducing the perpendicular to segment AB which contains point C, we get that triangles DBC and CBA are similar and triangles DAC and CAB are similar. So corresponding sides have lengths that are in the same proportion (this is Thales' Intercept Theorem, Theorem 1.1). Therefore

$$\frac{a}{p} = \frac{c}{a} \text{ which implies } a^2 = pc, \text{ and}$$

$$\frac{b}{q} = \frac{c}{b} \text{ which implies } b^2 = qc.$$

So $a^2 + b^2 = (pc) + (qc) = (p + q)c = c^2$ since $c = p + q$.

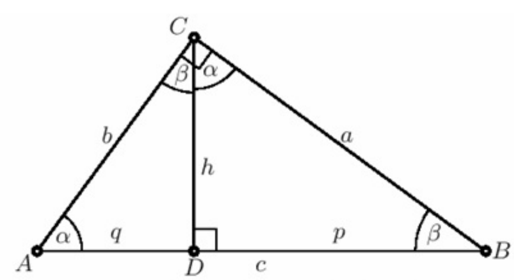


Fig. 1.20. A proof using Thales' theorem

This proof is credited to Leonardo of Pisa (also known as Fibonacci, circa 1170–1250) and was given in his *Practica Geometria* in 1200.

Note. We now turn out attention to the proof given by Pythagoras himself. Quoting from Sir Thomas Heath's *A History of Greek Mathematics*, Volume I, Clarendon Press (1921), which is still in print by Dover Publications and available for online reading from archive.org:

> "The next question is, how was the theorem proved by Pythagoras or the Pythagoreans? Vitrivius says that Pythagoras first discovered the triangle $(3, 4, 5)$, and doubtless the theorem was first suggested by the discovery that this triangle is right-angled; but this discovery probably came to Greece from Egypt. ...Two possible lines are suggested on which the general proof may have been developed. One is that of decomposing square and rectangular areas into squares, rectangles and triangles, and piecing them together again after the manner of Eucl., Book II; the isosceles right-angles triangle gives the most obvious case of this method. The other line is one depending upon proportions; and we have good reason for supposing that Pythagoras developed a theory of proportion. ...[Euclid] proved I.47 [the Pythagorean Theorem] by the methods of Book I instead of by proportions in order to get the proposition into Book I instead of Book VI [on proportions], to which it must have been relegated if the proof by proportions had been used. If, on the other hand, Pythagoras had proved it by means of the methods of Books I and II, it would hardly have been necessary for Euclid to devise a new proof of I.47. Hence it would appear most probably that Pythagoras would prove the proposition by means of his (imperfect) theory of proportions."

Note. For Pythagoras' proof by means of his "imperfect" theory of proportions, as Heath speculates, we consider the four shaded triangles in Figure 1.21. They have hypotheses of 1, a, b, and c, and they are similar to each other by construction. If k denotes the area of the triangle with hypotenuse 1, then the other triangles have the areas as given in the figure, namely ka^2, kb^2, and kc^2. This holds by Theorem 1.6 (which appears in the *Elements* as Euclid's VI.19). Comparing the center and left triangles, we see that $ka^2 + kb^2 = kc^2$, or $a^2 + b^2 = c^2$ as needed.

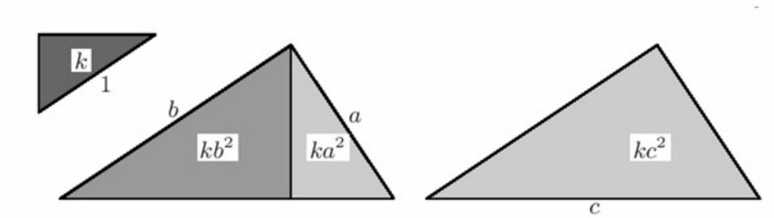


Fig. 1.21. Naber's proof

A quick comment is in order to Heath's use of the term "imperfect." The Pythagorean theory of proportion only applies to *commensurable* magnitudes. That is, it is only valid for rational proportions. This is suggested in Figure 1.6 (of Section 1.2. Similar Figures) where the technique of constructing rational links is given.

Note. We now use the Pythagorean Theorem to find the radii of the incircle and the circumcircle (denoted ρ and R, respectively) of a given regular n-gon with sides of length 1. In the case $n = 3$, these two circles are given in Figure 1.22 (left).

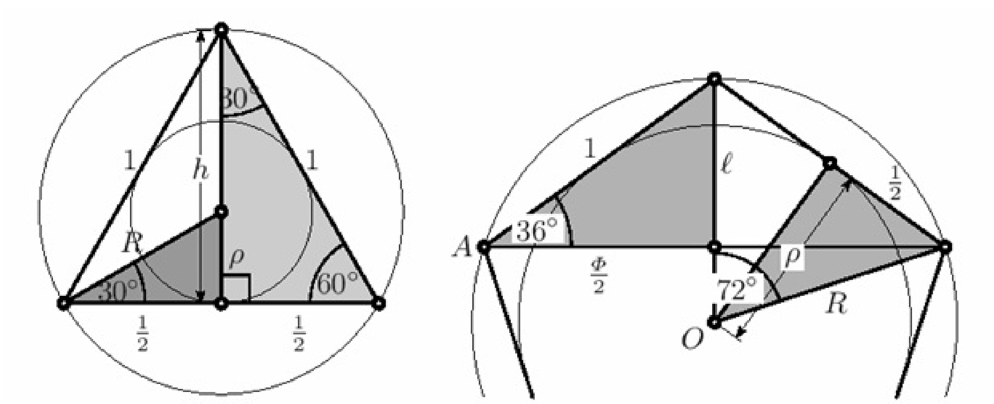


Fig. 1.22. Equilateral triangle and pentagon

In each case, we consider a line segment from the center of the circle to a vertex v of the n-gon; notice that such a line segment has length R. Next, we introduce a line segment from the center of the circle to the midpoint of one of the edges of the n-gon that has vertex v as one of its endpoints; notice that such a line segment has length ρ. The center of the circle, the vertex v, and the midpoint of the line segment then form a right triangle with sides of lengths r, ρ, and $1/2$ (the $1/2$ resulting from bisecting and edge of the n-gon; see Figure 1.22 for the cases of $n = 3$ and $n = 5$). So by The Pythagorean Theorem we have $R^2 = \rho^2 + (1/2)^2$, or $\rho = \sqrt{R^2 - 1/4}$. In the case $n = 3$, we introduce the distance h given in Figure 1.22 (left). We know, also by the Pythagorean Theorem, that $h = \sqrt{3}/2$ (since we are deal with a 30-60-90 triangle). Then

$$\rho = h - R = \sqrt{3}/2 - R \text{ and so } \sqrt{R^2 - 1/4} = \sqrt{3}/2 - R, \text{ or } R^2 - 1/4$$

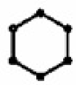
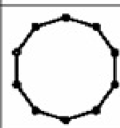
$$= 3/4 - \sqrt{3}R + R^2, \text{ or } R = 1/\sqrt{3} = \sqrt{3}/3.$$

Hence

$$\rho = \sqrt{(\sqrt{3}/3)^2 - 1/4} = \sqrt{1/3 - 1/4} = \sqrt{1/12} = 1/(2\sqrt{3}) = \sqrt{3}/6.$$

This is the first entry in Table 1.1 below. Similarly, in Figure 1,22 (right) we introduce the distance ℓ and we see that the larger shaded triangle has sides of lengths, 1, ℓ, and $\Phi/2$ (see the "golden ratio" in Section 1.4. The Regular Pentagon; recall that $\Phi^2 = \Phi + 1$). It is left as Exercise 1.7.A to show that R and ρ take on the values given in Table 1.1 in the case that $n = 5$.

Table 1.1. Radius of incircle (ρ) and radius of circumcircle (R) for regular polygons with side length 1

n		R	ρ
3		$R = \frac{\sqrt{3}}{3}$	$\rho = \frac{\sqrt{3}}{6}$
4		$R = \frac{\sqrt{2}}{2}$	$\rho = \frac{1}{2}$
5		$R = \frac{1}{\sqrt{3-\Phi}} = \frac{\sqrt{2+\Phi}}{\sqrt{5}}$	$\rho = \frac{\sqrt{3+4\Phi}}{2\sqrt{5}}$
6		$R = 1$	$\rho = \frac{\sqrt{3}}{2}$
10		$R = \Phi$	$\rho = \frac{\sqrt{3+4\Phi}}{2}$

Revised: 9/13/2021