Section 2.1. Book I.

Note. In this section (and the five that follow), we present the postulates, and some of the definitions and theorems of Euclid's *Elements*. Along the way, we give some commentary and a bit of criticism.

Note. Euclid's *Elements* starts unceremoniously with 23 definitions and no pictures. As a sampling of these definitions, we have:

Definition 1. A *point* is that which has no part.

Definition 4. A *straight line* is a line which lies evenly with the points on itself.

Definition 8. A *plane angle* is the inclination to one another of two straight lines in a plane which meet one another and do not lie in a straight line.

Definition 23. *Parallel straight lines* are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Of course this raises as many questions as it answers, since we now focus on the terms "part," "lies evenly," "inclination," and the meaning of "being produced indefinitely in both directions." Since we can only define new terms using old terms, at some point we must stop and simply take certain terms as undefined. The properties of these undefined terms are given to them by the postulates; see my online notes for "Introduction to Modern Geometry" (MATH 4157/5157) on Section 1.3. Axiomatic Systems. So this part of Euclid's approach is (by modern standards, at least) is futile!

Note. Book I contains five "postulates" (or assumptions). They are:

Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

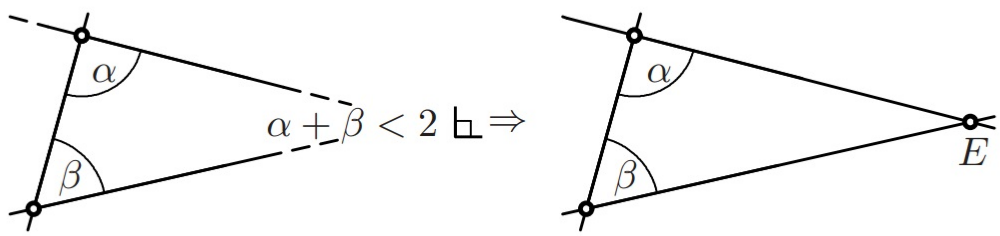
Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

The first three postulates are meant to insure the existence of certain constructions. Postulate 1 means that if two (distinct) points are given, then a line containing those two points can be constructed. Postulate 2 means that if a line segment (a "finite straight line") is given, then it can be extended to a (infinite, unbounded) line. Postulate 3 means that if a point is given and if a distance is given (in terms a particular line segment), then a circle with the point as its center and the distance as its radius can be constructed. This terminology is used throughout the *Elements* (along with an unusual way of distinguishing between lines and line segments). Postulate 4 claims an equality of a certain class of angles; it is actually the measure of the angles that are being claimed to be equal (though the measure of an angle is never defined). Ostermann and Wanner state that Postulate 4 "expresses the homogeneity of space in all directions" (see their page 30); this is usually called the property of *isotropy*. Notice that the first four postulates are unsurprising and uncomplicated. However, Postulate 5 could use some additional exploration.

Note. Postulate 5 is the *Parallel Postulate*. Think of the "two straight lines" as being given, and then a "straight line falling" on these as a transversal cutting both lines. The idea of "interior angles" requires some concept of "betweenness" (another shortcoming of Euclid's approach; see my online notes for "Introduction to Modern Geometry" [MATH 4157/5157] on Section 2.5. Order Relations). The condition "less than two right angles" requires (again) the idea of a measure of an angle (and, if we are being picky, the "side" of a line is never defined). So by our 21st century standards, Euclid lacks some rigor. But, of course, there is good stuff here and we continue! Ostermann and Wanner illustrate the Parallel Postulate as follows (with angles α and β as the two interior angles):



We comment in passing that the complicated nature of the Parallel Postulate lead some to try to *prove* it based on other theorems. None of these approaches were successful, but some lead to the discovery of non-Euclidean geometry.

Note. Book I also contains five "Common Notions." These are related to arithmetic relationships concerning equality and "greater than." The common notions are:

Common Notion 1. Things which equal the same thing also equal one another.

Common Notion 2. If equals are added to equals, then the wholes are equal.

Common Notion 3. If equals are subtracted from equals, then the remainders are equal.

Common Notion 4. Things which coincide with one another equal one another.

Common Notion 5. The whole is greater than the part.

Note. We now turn our attention to the propositions. Book I contains 48 propositions which address the construction of equilateral triangles, certain lines, congruent triangles, properties of triangles, bisection of an angle and a line segment, angles determined by two intersecting lines and angles in a triangle, interior and alternate angles and their relationship to parallel lines, parallelograms, and the Pythagorean Theorem.

Euclid, Book I Proposition 1. On a given finite straight line AB to construct an equilateral triangle.

Note. The existence of the point of intersection Γ of the two circles in the proof of Proposition 1 reveals another weakness in Euclid's approach. This requires some concept of continuity and a continuum (Heath comments in his translation of the *Elements* that both Zeno and Proclus brought attention to this; see pages 242 and 243). Euclid deserves a real break on this, since these ideas were not made completely rigorous until the mid 1800's. For additional discussion, see my online notes for "Introduction to Modern Geometry" [MATH 4157/5157] on Section 2.4. The Measurements of Distance.

Euclid, Book I Proposition 2. To place at a given point A a straight line AE equal to a given straight line $B\Gamma$.

Note. The relative positions of points A, B, and Γ in the proof of Euclid I.2 requires the consideration of cases (the arguments being similar in each case). This is an objection also raised by Proclus (see Heath's *Elements*, page 245). The *reason* that Euclid I.2 is proved is that it allows us to construct at any point A a circle with a radius given by some other line segment (as opposed to Postulate 3, which only allows the creation of a circle centered at a given point and passing through a given point). The idea is that we can set a compass at a certain radius and then "compass-carry" that radius to any given point. As usual, the *Elements* are written with an eye towards compass and straightedge constructions.

Euclid, Book I Proposition 4. Given two triangles with $a = a'$, $b = b'$, $\gamma = \gamma'$, then all sides and angles are equal.

Note. Euclid I.4 is commonly called "Side-Angle-Side" (SAS) and it gives a condition under which two triangles are congruent. Euclid's proof involves an idea of superposition (as does his Common Notion 4 when he uses the term "coincides"). This approach is often criticized; Euclid has not addressed any type of motion in his definitions, postulate, or common notions. See Heath's *Element's*, page 249. These ideas of motion and superposition can be dealt with using transformations and these ideas are preserved in a modern geometry class by considering transformational geometry (see my online notes for Introduction to Modern Geometry [MATH 4157/5157] for Transformational Geometry). It is also common in a modern geometry class to take Euclid I.4 as a postulate. This is done by David Hilbert in his 1899 *The Foundations of Geometry* which was an attempt to clearly state all the postulates (or "axioms") of Euclidean geometry and to present it in a rigorous, purely mathematical way (without, for example, an appeal to pictures). We'll consider Hilbert's work again briefly in Section 2.7. Epilogue. In my online notes for Introduction to Modern Geometry [MATH 4157/5157] on Section 2.9. The Congruence Postulate (see Postulate 16).

Note. Euclid states his Proposition 5 as: "In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another." Ostermann and Wanner state it more algebraically and use the notation that the length of a side of a triangle is expressed using lower case English letters and the measure of an angle opposite a side is labeled with the corresponding lower case Greek letter (so side a is opposite angle α, for example).

Euclid, Book I Proposition 5. If in a triangle, $a = b$, then $\alpha = \beta$.

Note. Euclid I.5 is sometimes called (and this is mentioned by Ostermann and Wanner) the *pons asinorum* or asses' bridge. This is addressed in David Eugene Smith's *History of Mathematics, Volume II*, Ginn and Company (1925). He even titles a subsection of Section V.5 "Pons Asinorum." On page 284 he comments:

> "The proposition represented substantially the limit of instruction in many courses in the Middle Ages. It formed a bridge across which fools could not hope to pass, and was therefore known as the *pons asinorum*, or bridge of fools. It has also been suggested that the figure given by Euclid resembles the simplest form of a truss bridge, one that even a fool could make. The name seems to be medieval."

So Proposition 5 marks a transition from elementary results to more complicated results.

Note. Notice in the proof of Euclid I.5, by applying Postulate 2 and "continuously" extending line segments, we are introducing the concept of continuity and using it intuitively. In a modern approach to geometry, we would deal with the continuity concept with a postulate that effectively treats every line in Euclidean geometry with the real number line. See my online notes for Introduction to Modern Geometry (MATH 4157/5157), the Axiomatic Method, on Section 2.4. The Measurements of Distance; notice The Ruler Postulate, Postulate 11. In this way, all continuity concerns are pushed onto to the real number line and are dealt with in a senior-level analysis class; see my online notes for Analysis 1 (MATH 4217/5217) on Section 1.3.

The Completeness Axiom. This may also be addressed in a freshman-level calculus class. See my online Calculus 1 (MATH 1910) notes on Appendix A.6. Theory of the Real Numbers for an informal discussion of a rigorous approach to continuity of the real line (a video for this section is also available). Interestingly, continuity of the real line (an idea implicit to any study of geometry, including Euclid's 23000 year-old *Elements*) is not cleanly defined until 1872 by Richard Dedekind in his "Continuity and Irrational Numbers," a copy of which can be found at Project Gutenberg.

Note. Pappus of Alexandria (circa 290–circa 350), presumably in a commentary on Euclid's *Elements*, gave an elementary proof of Euclid I.5 based on Euclid I.4 (we'll discuss Pappus in more detail in Section 4.1. The Conchoid of Nicomedes, The Trisection of an Angle). He argues that triangles ACB and BCA are equal by Euclid I.4 (side-angle-side), so that the corresponding angles α and β must be equal, as claimed; see Figure 2.2(b) and (c).

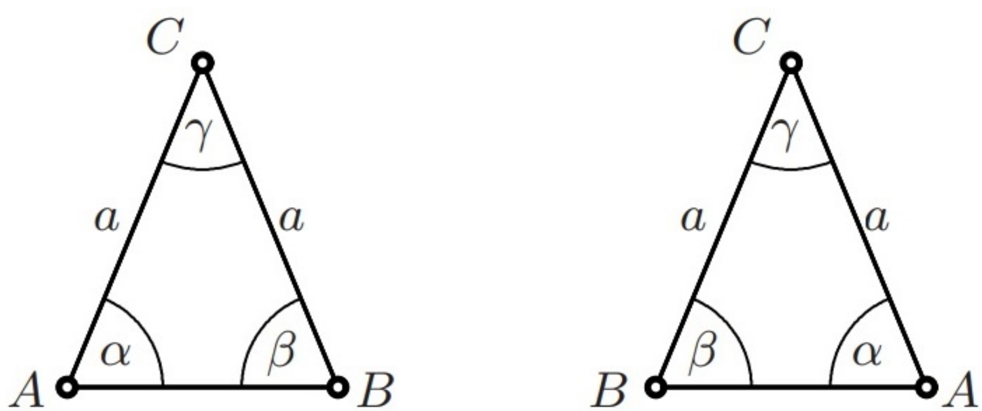


Figure 2.2(b) and (c)

Note. Euclid I.6 is the converse of Euclid I.5. That is, in the notation of Euclid I.5, if $\alpha = \beta$ then $a = b$. The next two results concern the equivalence of triangles based on the lengths of the three sides.

Euclid, Book I Proposition 7. Consider the two triangles of Figure 2.3(a), with the same base AB and with the third vertex on the same side of the base. If $a = a'$ and $b = b'$, then points C and D are the same, $C = D$.

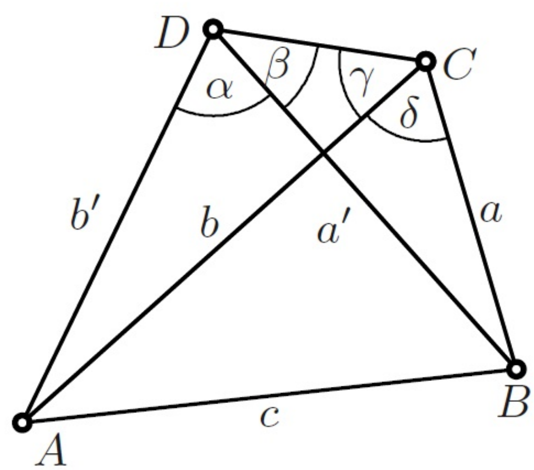


Figure 2.3(a)

Note. Euclid's proof of Euclid I.7 is an "indirect proof" or a "proof by contradiction." Philosophers call this type of argument (in Latin) *reductio ad absurdum*. See my online notes for Mathematical Reasoning (MATH 3000) on Section 1.4. Proofs: Structures and Strategies for more details.

Euclid, Book I Proposition 8. If two triangles ABC and DEF have sides of equal lengths, then they also have equal angles.

Note. The proof for Euclid I.8 of Philo of Byzantium given in the supplement involves an idea of movement when one line segment is placed on another line segment of the same length so that the respective endpoints of the two line segments coincide. Though certainly intuitively clear, nothing in the postulates nor the common notions address movement (though Common Notion 4 mentions "things which coincide"). Movement can be rigorously addressed by introducing transformations. This falls in the realm of *transformational geometry*. For this approach, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on transformational geometry, where the complex plane is the setting for Euclidean geometry (and a subset of it is the setting for the non-Euclidean geometry called hyperbolic geometry).

Note. Euclid's Propositions 9–12 in Book I are illustrated in Figure 2.4 below. Euclid I.9 gives a construction for bisecting a given angle (the idea with these constructions is that they are performed with a compass and straightedge). Euclid I.10 gives the construction for the bisection of a line segment. Euclid I.11 gives a construction of a perpendicular to a line through a given point on the line. These results use an equilateral triangle, which we know to exist by Euclid I.1. Euclid I.12 gives a construction of a perpendicular to a line through a given point *not* on the line; this result uses a circle (the idea that the circle is constructed with a compass; of course, a circle was used in the proof of Euclid I.1 as well).

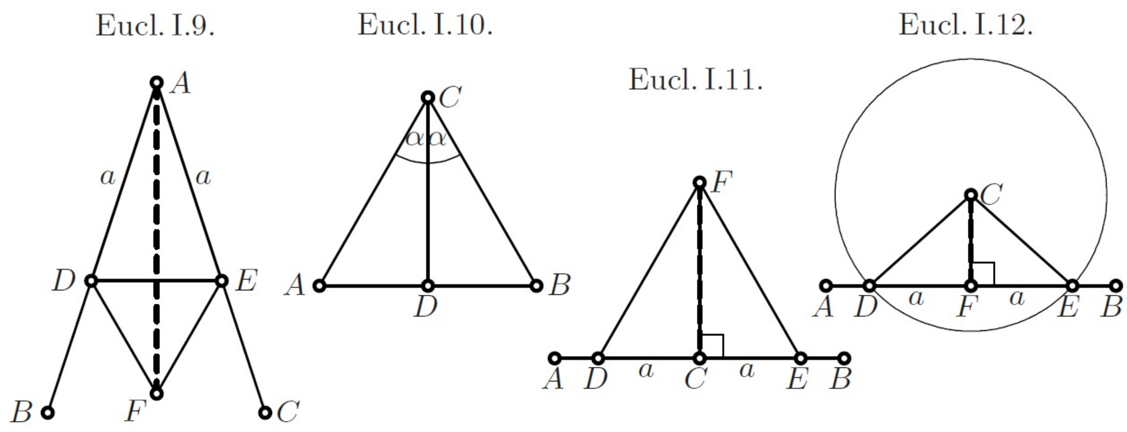


Figure 2.4. Propositions I.9–I.12

Note. Recall that Postulate 4 states: "That all right angles equal one another." Ostermann and Wanner explain the significance of Postulate 4 (see pages 33 and 34) as:

> "The fourth postulate expressed the homogeneity of the plane, the absence of any privileged direction, and allows one to compare, add and subtract the angles around a point."

Sir Thomas Heath makes a similar comment in his translation of the *Elements* (see page 200 of the second edition of his volume 1; his emphasis):

> "While this Postulate asserts essential truth that a right angle is a *determinate magnitude* so that it really serves as an invariable standard by which other (acute and obtuse) angles may be measured. . . If the statement [i.e., Postulate 4] is to be *proved*, it can only be proved by the method of applying one pair of right angles to another and so arguing their equality. But this method would not be valid unless on the assumption of the *invariability of figures*, which would therefore

> have to be asserted as an antecedent postulate. Euclid preferred to assert as a postulate, directly, the fact that all right angles are equal. . . and hence his postulate must be taken as equivalent to the principle of *invariability of figures* or its equivalent, the *homogeneity of space*."

This idea of homogeneity as a property of geometry even holds in the non-Euclidean cases. To consider a situation where a space is *not* homogeneous, is not to do geometry globally! However, "locally" geometry can be (approximately) addressed in a nonhomogeneous space. This is the realm of differential geometry, where the curvature can vary with location in the space. For example, a sphere is homogeneous and has the same (positive) curvature at all points. However, a torus is not homogeneous and has different curvature (sometimes positive, sometimes negative, and sometimes zero) at different points. See my online notes for Differential Geometry (MATH 5310); I also have some more rigorous notes on differential geometry available.

Note. We denote a right angle, that is a 90° angle, by the symbol ⊾. When dealing with a 180° angle, Euclid explicitly refers to the amount "two right angles." Sometimes this is called a "straight angle." The next proposition concerns this idea.

Euclid, Book I Proposition 13. Let the line AB cut the line CD. With α and β as the two resulting angles on the same side of line CD, we have $\alpha + \beta = 2$⊾.

Euclid, Book I Proposition 14. Let line segment DB and line segment BA determine an angle β. If segment BC makes an angle α with segment BA where point C is exterior to the first angle. If $\alpha + \beta = 2 ⊾$ then C lies on the line DB. See Figure 2.6.

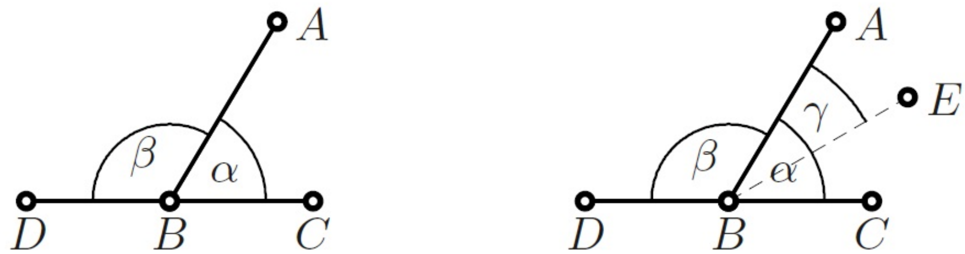


Figure 2.6

Euclid, Book I Proposition 15. If two lines cut one another, they make the opposite angles equal to one another.

Euclid, Book I Proposition 16. If one side of a triangle is extended at C, the exterior angle is greater than both angles in the triangle opposite to C.

Note. Figure 2.9 is synopsis of Euclid's results concerning equal triangles. As we have seen, Euclid I.4 says that is two sides and an included angle are equal between two triangles (i.e., side-angle-side of SAS) then the triangles are equal. Euclid I.8 says that if three corresponding sides of two triangles are equal (i.e., side-side-side or SSS) then the triangles are equal. Euclid I.26 states: "If two triangles have two angles equal to two angles respectively, and one side equal to one side, namely, either the side adjoining the equal angles, or that opposite one of the equal angles, then the remaining sides equal the remaining sides and the remaining angle equals the remaining angle." That is, we have equal triangles under angle-side-angle (or ASA) and angle-angle-side (or AAS), as illustrated in Figure 2.9 in separate cases. Ostermann and Wanner feel compelled to emphasize that there is no angle-side-side (or ASS).

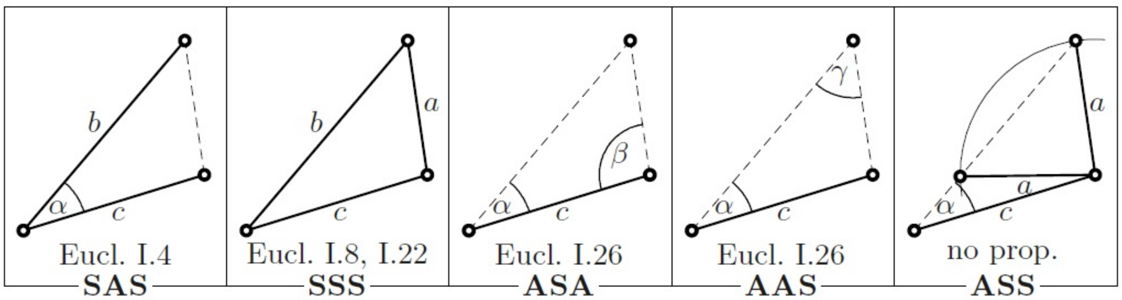


Figure 2.9

Note. Euclid's Book I Proposition 22 states: "In any triangle the sum of any two sides is greater than the remaining one." In our symbols, this is the three claims:

$$a < b + c, \ b < c + a, \quad \text{and} \quad c < a + b.$$

A proof of Euclid I.22 is to be given in Exercise 2.12. This is a fundamental result and is called *The Triangle Inequality*. You will encounter this in your future math classes whenever dealing a measure of distance. Distance is measured, in general, with a *metric* and a metric, by definition, must satisfy the triangle inequality. You also see a mention of this in connection with norms in a vector space. For more details, see my online notes for Linear Algebra (MATH 2010) on Section 1.2. The Norm and Dot Product (where the Triangle Inequality is addressed in for the norm on the vector space $\mathbb{R}^n$) and Section 3.5. Inner-Product Spaces (where the Triangle

Inequality is addressed for theorem induced by an inner product in an abstract vector space). For more on the Triangle Inequality and metrics, see my online notes for Introduction to Topology (MATH 4357/5357) on Section 20. The Metric Topology and my online notes for Complex Analysis 1 (MATH 5510) on Section II.1. Definitions and Examples of Metric Spaces.

Note. We now turn our attention to parallel lines. First, recall that parallel lines are lines that do not intersect or, as Euclid puts it:

> **Definition 23.** *Parallel straight lines* are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

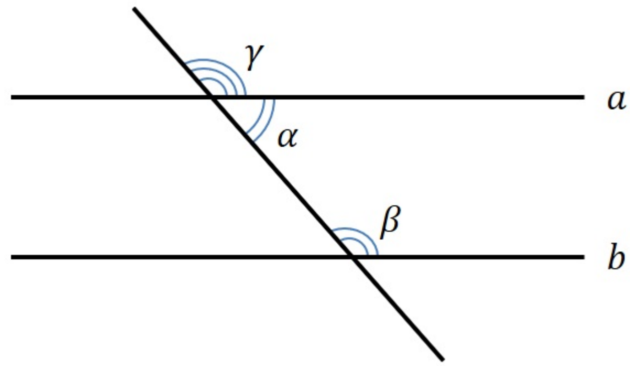
The next result addresses the existence of parallel lines.

Euclid, Book I Proposition 27. If some line cuts two line a and b such that alternate interior angles α and β are equal, then lines a and b are parallel, denoted $a \parallel b$.

Note. Euclid's last result that does not use the Parallel Postulate is the following:

> **Euclid, Book I Proposition 28.** If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.

That is, if $\gamma = \beta$ or if $\alpha + \beta = 2 \llcorner$ in the following figure, then lines $a \parallel b$.



Notice that the conclusions of both Euclid I.27 and I.28 are "then the lines are parallel." Going in the other direction starting with parallel line and drawing conclusions about angles requires the Parallel Postulate (Postulate 5).

Note. None of the results in Book I, up through Proposition 28, use Postulate 5 (The Parallel Postulate). It seems that Euclid is postponing the use of Postulate 5 as long as possible. Notice the clunky nature of the statement of Postulate 5 as compared to the other postulates and the common notions. Geometry based on the first four postulates of Euclid (more accurately, a geometry based on an axiomatic systems which excludes Postulate 5) is called *absolute geometry* or *neutral geometry*. Both Euclidean geometry and the version of non-Euclidean geometry called hyperbolic geometry are examples of neutral geometry. The version of non-Euclidean geometry called elliptic geometry (a special case of which is spherical geometry) does not include Euclid I.16 (an exterior angle of a triangle at point C is greater than both angles in the triangle opposite to C) and so is not an example of neutral geometry (Ostermann and Wanner mention this in passing on page 35). Since the proof of the existence of parallel lines in Euclid I.27 uses Euclid I.16, then we may not have the existence of parallel lines in elliptic geometry (and, in fact, parallel lines do not exist in elliptic geometry. For a brief discussion of this, see my online presentation on A Quick Introduction to Non-Euclidean Geometry. Recall the statement of the Parallel Postulate:

> **Postulate 5, The Parallel Postulate.** That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Note. Euclid's first result which requires the use of the Parallel Postulate is the following: Euclid's next result is a converse of Euclid I.27, as follows.

Euclid, Book I Proposition 29. Parallel lines cut by some line, have alternate interior angles are equal.

Note. For about 2000 years, attempts were made to either prove the Parallel Postulate or prove something like Euclid I.29 without using the Parallel Postulate. After all, these are intuitively obvious claims! But what makes them "obvious" is our prejudice for Euclidean geometry based on our experience and intuition. These attempts led to consideration of the angles in a quadrilateral. In particular, if we consider a quadrilateral with base angles as right angles and sides rising from the base of the same length then, in Euclidean geometry, the two "summit" angles are right angles (and *if* the summit angles are right angles, then the Parallel Postulate can be proved). Such a quadrilateral is called a Saccheri quadrilateral (after the work of Giovanni Girolamo Saccheri, September 5, 1667–October 25, 1733). Three 19th century mathematicians are credited with founding the version of the non-Euclidean geometry called hyperbolic geometry. The are Carl Frederick Gauss (1777–1855), Nicolai Lobachevsky (1793–1856), and Johann Bolyai (1802–1860). Letters of correspondence by Gauss indicate that he came to started to work on hyperbolic geometry in the 1790s and had developed fundamental theorems in this new geometry some time shortly after 1813. However, he never published his results. Lobachevsky (a Russian) published his "On the Principles of Geometry" in the *Kasan Bulletin* in 1829-30 and so was the first to publish results on hyperbolic geometry. This is recognized today in that hyperbolic geometry is sometimes called "Lobachevskian geometry." However, Lobachevsky's work gained little attention (and since it was published in Russian, it was not widely circulated in western Europe; he would later publish his work in German and French). Bolyai, unaware of the work of Gauss or Lobachevsky, in 1832 published "The Science of Absolute Space" on hyperbolic geometry as an appendix to a book that his father authored. There was controversy as to who deserved credit for being the first to consider hyperbolic geometry. A simplified version of the history is that Gauss was first (but never published), Lobachevsky was the first to publish but may have been influenced by some of Gauss's ideas (they had corresponded). Bolyai seems to have worked independently, but he seems to remain "number 3" in this trio in terms of credit. For a more detailed discussion (and references), see my online presentation on Hyperbolic Geometry.

Carl Frederick Gauss — Nicolai Lobachevsky — Johann Bolyai

4/20/1777–2/23/1855 — 12/1/1793–2/24/1856 — 12/15/1802–1/27/1860

Images from the MacTutor History of Mathematics Archive (accessed 1/19/2022)

This type of event where multiple mathematicians have similar ideas at the same time (they are not working in a mathematical vacuum) is not unprecedented. This occurred in the 1530s and 1540s when Tartaglia and Gerolamo Cardano argued over priority of the quartic equation (sort of a quadratic formula for 4th degree polynomials) and again around 1700 with the work of Newton and Leibniz on calculus. In these cases (as also, to a degree, with Gauss) arguments, bitterness, and unprofessional behavior are part of the history.

Note. The next result shows that the relationship of "parallel" between lines is transitive.

Euclid, Book I Proposition 30. For any three (distinct) lines a, b, c, if $a \parallel b$ and $b \parallel a$ then $a \parallel c$.

Euclid, Book I Proposition 31. To draw a parallel to a given line a through a given point A not on the a.

Note. We mentioned Proclus' *Commentary* on Euclid's *Elements* in Introduction to Chapter 2. In his "Propositions: Part Two" he comments that in Euclid I.31 there cannot be two lines through point A parallel to line a (see Proclus' page 376). He gives the brief argument that if there were two such lines then the two lines would be parallel to each other (by Euclid I.30), but then there would be two parallel lines which intersect at point A, a contradiction. This result is often called Playfair's Axiom because it appeared in John Playfair's *Elements of Geometry* in 1795. Playfair states his axiom as:

> **Axiom 11.** Two straight lines which intersect one another, cannot be both parallel to the same straight line.

An 1846 version of Playfair's book is online at Archive.org (accessed 1/20/2022). If we take Playfair's Axiom that there is exactly one line parallel to line a through point A, then we can derive Euclid's Parallel Postulate as a theorem. The appeal of Playfair's approach is that it is easy to negate Playfair's Axiom (and hence to negate Euclid's Parallel Postulate). We see that we have two choices for the negation: (1) there are no lines through A parallel to a, or (2) there are two (or more) lines through A parallel to a. Each then leads to two versions of non-Euclidean geometry: (1) elliptic geometry, and (2) hyperbolic geometry, respectively. This is the approach taken in my online presentation: A Quick Introduction to Non-Euclidean Geometry. Sometimes in a high school geometry class, some version of Playfair's Axiom is used in place of Euclid's Parallel Postulate.

Note. Book I Proposition 32 states that the sum of the angles of a triangle is equal to two right angles: $\alpha+\beta+\gamma = 2\llcorner$. Ostermann and Wanner state (see page

38): "This is a very old theorem, certainly known to Thales." The proof ultimately depends on Euclid's Parallel Postulate. Interestingly, in elliptic geometry the sum of the angles of a triangle is greater than $2\,\llcorner$, and in hyperbolic geometry the sum of the angles of a triangle is less than $2\,\llcorner$. Surprisingly, in non-Euclidean geometry the angle sum depends on the *size* of the triangle (implying some fundamental unit of length in these geometries).

Note. The remainder of Book I addresses parallelograms, areas of parallelograms and triangles, the Pythagorean Theorem Proposition 47), and its converse (Proposition 48, the final proposition in Book I). We considered Euclid's proof of the Pythagorean Theorem in our Section 1.7. The Pythagorean Theorem.

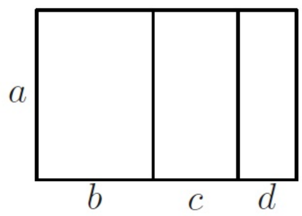
Note. Book II contains geometrical algebra. That is, algebra expressed in geometric terms. Some of our algebraic terms today, such as "squaring" and "cubing," reflect this interpretation. Book II is much shorter than Book I, only containing 14 propositions. Our current algebraic notation dates from much after the time of Euclid. However, in this discussion we express Euclid's results in modern notation and give corresponding geometric figures. Euclid II.1 states:

> **Book II, Proposition 1.** If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments.

Euclid II.4 states:

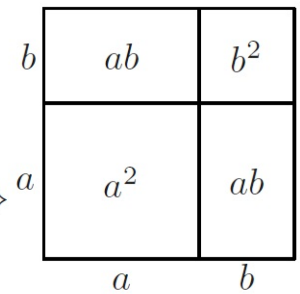
Book II, Proposition 4. If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments.

These propositions are illustrated in the following diagrams (from page 38):



$$\Leftrightarrow \quad a(b+c+d) = ab + ac + ad$$

$$(a+b)^2 = a^2 + 2ab + b^2 \quad \Leftrightarrow$$



Note. Euclid II.5 geometrically proves the algebraic relationship $a^2 - b^2 = (a+b)(a-b)$. See Figure 2.14 (left). Euclid II.13 geometrically proves that $2uc = b^2 + c^2 - a^2$ where segment u results from an altitude from a vertex of a triangle to the opposite side, as shown in Figure 2.14 (middle); Euclid's proof employs the Pythagorean Theorem.

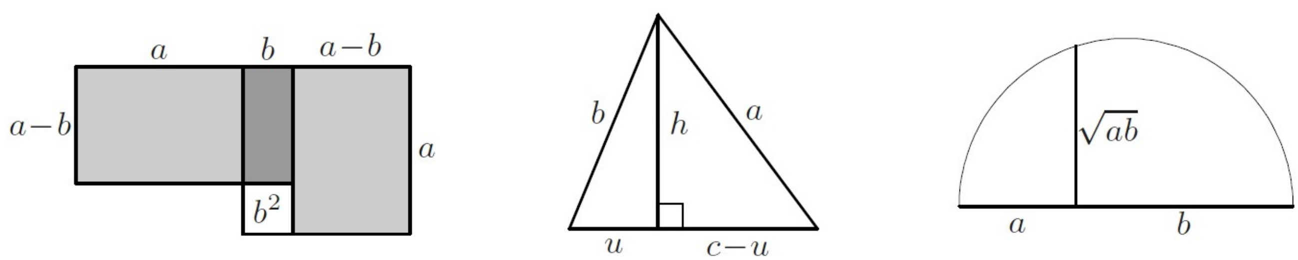


Figure 2.14