Section 2.2. Book III. Properties of Circles and Angles

Note. Book III of the *Elements* has 37 propositions. In this section we consider a few of these propositions, mostly those related to angles and chords related to a circle.

Note. We have already seen some results concerning circles in Section 1.3. Properties of Angles. In particular, we had:

Theorem 1.4 (Euclid III.20, The Central Angle Theorem). A central angle of a circle is twice any inscribed angle on the same arc. See Figure 1.9(c) in Section 1.3.

Theorem 1.5. (Euclid III.31). If AB is the diameter and C is a point (other than A or B) on the circle, then angle ACB is a right angle. See Figure 1.9(d) in Section 1.3.

A related result is Euclid's Book III Proposition 21, a proof of which is to be given in Exercise 1.3:

Exercise 1.3. (Euclid III.21). Let ABC be a triangle inscribed in a circle. See Figure 1.25 (right). The size of angle α is independent of the position of A on the circle.

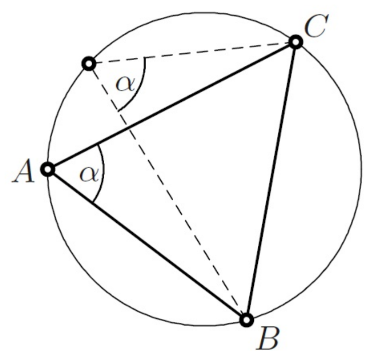

Figure 1.25 (right)

Note. We now state and prove several results from Book III, mostly concerning angles and chords related to a circle.

Euclid, Book III Proposition 22. Let $ABCD$ be a quadrilateral inscribed in a circle (see Figure 2.15(a)). Then the sum of two opposite angles equals two right angles: $\alpha + \delta = 2\llcorner$.

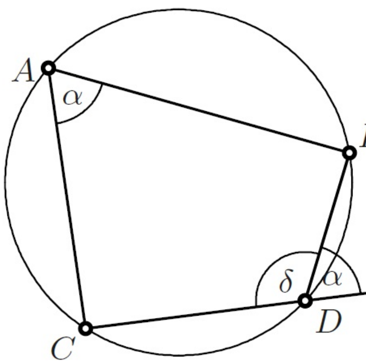

Figure 2.15(a)

Euclid, Book III Proposition 35. If two chords AB and CD of a circle intersect in a point E inside the circle (see Figure 2.16(a)), then $AE \cdot EB = CE \cdot ED$.

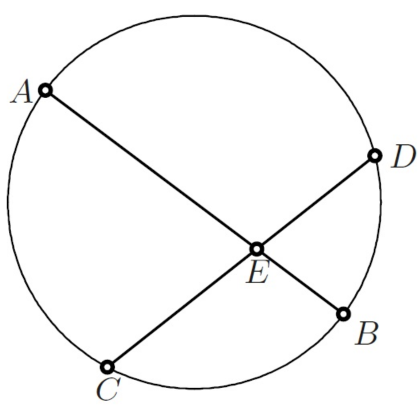

Figure 2.16(a)

Euclid, Book III Proposition 36. Let E be a point outside a circle and consider a line through E that cuts the circle in two points A and B. Further let T be the point of tangency of a tangent through E (see Figure 2.17(a)). Then $AE \cdot BE = (TE)^2$.

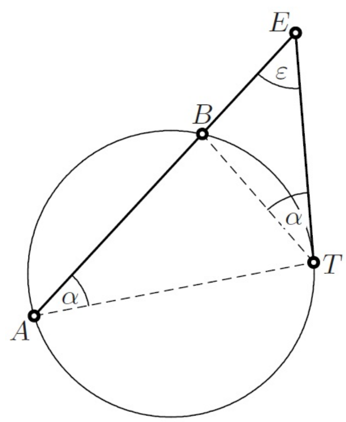

Figure 2.17(a)

Corollary (Clavius 1574). Let A, B, C, and D denote four points on a circle. If the line AB meets the line CD in a point E outside the circle (see Figure 2.17(b)), then $AE \cdot BE = CE \cdot DE$.

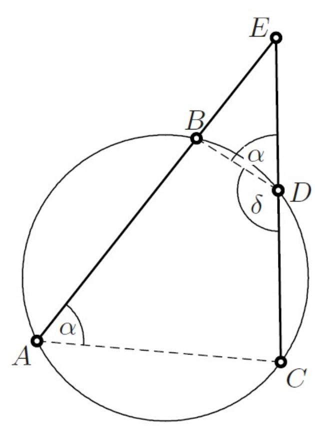

Figure 2.17(b)

Note/Definition. In Euclid III.36 and Figure 2.17(a), if AB is a diameter of the circle then, with distances as labeled in Figure 2.17(c), we have $t^2 = (d+r)(d-r) =$

$d^2 - r^2$. Notice that this also follows by the Pythagorean Theorem (Euclid I.47) since the angle at point T is a right angle (by Euclid III.18; see Exercise 2.16). The quantity $d^2 - r^2$ is the *power of the point E with respect to the circle.* This is related to the idea of points "symmetric with respect to a circle." See my online notes for Complex Analysis 1 on Section III.3. Analytic Functions as Mapping, Mobius Transformations; notice the Symmetry Principle (Theorem III.3.19) and the Note before it.

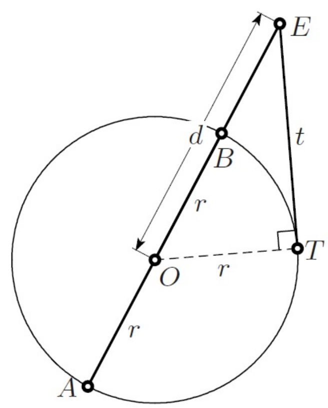

Figure 2.17(c)

Note. Book IV of the *Elements* has 16 propositions. Most deal with circles inscribed in or circumscribed by triangles, square, regular pentagons, and regular hexagons. Proposition 16 considers the construction of a regular 15-sided polygon in the setting of inscribing it in a circle. We'll see in Section 5.10. The Great Discoveries of Kepler and Newton that Johannes Kepler (December 27, 1571-November 15, 1630) tried (unsuccessfully) to describe the location of the planets in the solar system by using ideas similar to those of Book IV, except that he did his work in three dimensions. He considered spheres embedded in regular polyhedra.