Section 2.3. Books V and VI. Real Numbers and Thales' Theorem

Note. In this section we briefly consider the content of Book V (in which a theory of proportions and "incommensurables" is developed) and Book VI (which involves applications of the theory of proportions).

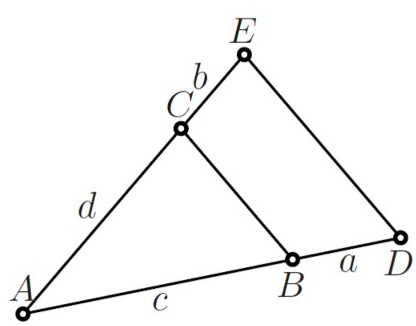
Note. Book V of the *Elements* has 25 propositions. Ostermann and Wanner only devote five lines to it (arguably, Book V is more number theory than geometry). Heath mentions in his *The Thirteen Books of Euclids Elements* (Cambridge University Press, 1926) that the content of Book V is the discovery of Eudoxus, the teacher of Plato (Heath references a source perhaps, due to Proclus). It is on the theory of proportions. Heath observes that the theory of proportions is covered twice in the *Elements*, in Book V in reference to "magnitudes" and in Book VII in reference to "numbers." Book VII covers only "commensurables"; that is, positive rational numbers. Euclid doesn't define "magnitude," but it refers to an amount of something, such as the length of a line segment, area, volume, or an amount of time. It seems (this is your humble instructor's view here) that a magnitude is an attempt to consider a continuum.

Note. Book VI involves applications of the theory of proportions developed in Book V. There are 33 propositions in Book VI. It is commented on David Joyce's (of Clark University) online version of the *Elements* that: "Proposition VI.1 is the basis for the entire of Book VI except the last proposition VI.33." (Proposition VI.33 concerns angles in circles, whereas the other propositions concern triangles, parallelograms, and rectilinear figures [i.e., polygons].) Euclid VI.1 concerns the ratio of the areas of two triangles (or two parallelograms) and states that if the heights are the same, the the ratio of the areas is the same as the ratio of the bases:

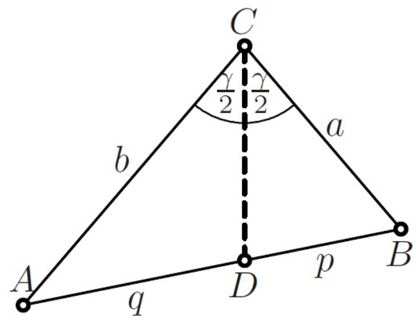
> **Euclid, Book VI Proposition 1.** Triangles and parallelograms which are under the same height are to one another as their bases.

Note. We have taken Thales' Theorem (Theorem 1.1) concerning similar triangles as true when stated in Section 1.1. Thales' Theorem. We have presented a few proofs using Thales's Theorem (see Euclid III.35 and III.36 in the previous section), but Euclid does not give a proof of Thales' Theorem until Book VI (so our proofs of Euclid III.35 and III.36 are different from Euclid's). Euclid presents Thales' Theorem as follows.

Euclid, Book VI Proposition 2. Consider triangle ADE. Suppose B is a point on segment AD and C is a point on segment AE such that BC is parallel to DE. Then $a/c = b/d$ (where a, b, c, d are the distances given in the figure).



Euclid, Book VI Proposition 3. Consider triangle ABC with angle γ at point C. Let CD be the bisector of γ. Then $a/b = p/q$ (where a, b, p, q are the distances given in the figure).



Note. Ostermann and Wanner describe the other propositions in Book VI as "variants of Thales' theorem and their converses." See page 43. That is, the propositions deal with proportions between lengths and areas of triangles, parallelograms, and polygons. In particular, Euclid VI.9 gives a technique to subdivide a line segment into equal-length parts. So if we start with a length 1 line segment (or we use a given line segment to define "length 1") then we can construct segments of lengths $1/n$ for any $n \in \mathbb{N}$. This then allows us to construct segments of any rational length m/n. We say that the rational numbers are constructible. But the rational numbers are not the only constructible numbers, since the Pythagorean Theorem and the construction of right triangles implies that segments of length $\sqrt{2}$ (for example) are constructible. Exactly which real numbers are constructible is dealt with in Introduction to Modern Algebra 2 (MATH 4127/5127) in Section VI.32. Geometric Constructions; notice Corollary 32.5 and Theorem 32.6.

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