Section 2.5. Book XI. Spatial Geometry and Solids

Note. In this section we move out of the plane and into the third dimension. We give a few definitions, and consider several solids which we simply illustrate with figures. We consider the Platonic solids and their dual relationships.

Note. Book XI has 28 definitions and 39 propositions. It concerns geometry in 3-dimensions. A few of the definitions are:

Book XI, Definition 1. A *solid* is that which has length, breadth, and depth.

Book XI, Definition 2. A face of a solid is a surface.

Book XI, Definition 3. A straight line is *at right angles* to a plane when it makes right angles with all the straight lines which meet it and are in the plane.

Book XI, Definition 8. *Parallel planes* are those which do not meet.

We address several of the other definitions by considering pictures.

Note. We follow the wording and images of Ostermann and Wanner. A *pyramid* is a solid formed be a polygon, an apex point, and triangles (see Figure 2.22).

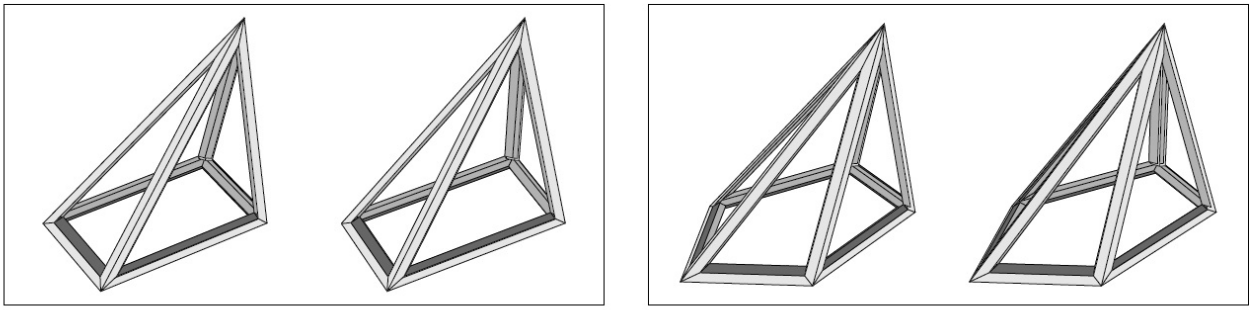


Figure 2.22. Pyramids over a rectangle and a pentagon

A *prism* is a solid formed by a polygon in a plane, a second identical polygon in a plane parallel to the first plane, and parallelograms (see Figure 2.23).

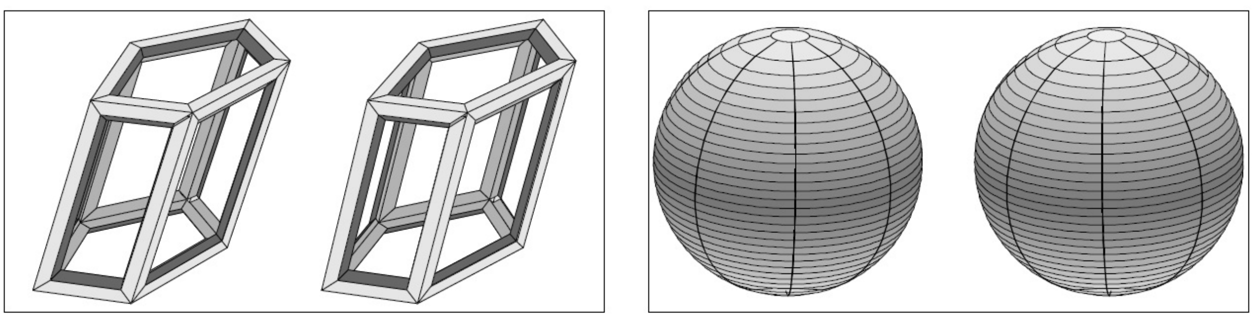


Figure 2.23. A prism over a pentagon (left)

A *cone* is a solid that results by rotating a right-angled triangle around a leg (see Figure 2.24, left). A *cylinder* is a solid that results by rotating a rectangle around a side (see Figure 2.24, right).

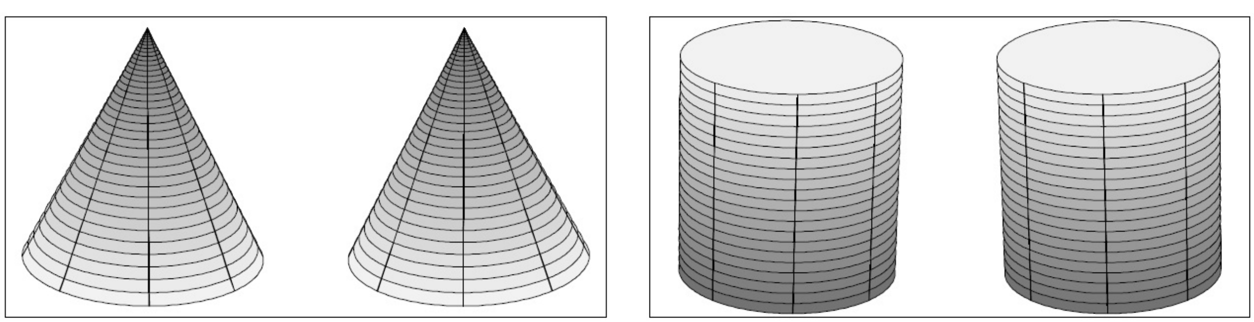


Figure 2.24. Cones and cylinders

Note. We now consider the five platonic solids. Each has faces which are regular polygons (that is, polygons with all sides the same length and all angles equal). A *tetrahedron* is a pyramid with an equilateral triangle as the base and each of the other three faces also equilateral triangles, as shown in Figure 2.28 left (below). A *cube* is a prism with base a square, a second square as required be the definition of a prism, and the remaining faces also squares (see Figure 2.25 left). An *octahedron* is an 8-faced solid where each face is an equilateral triangle, as shown in Figure 2.25 right.

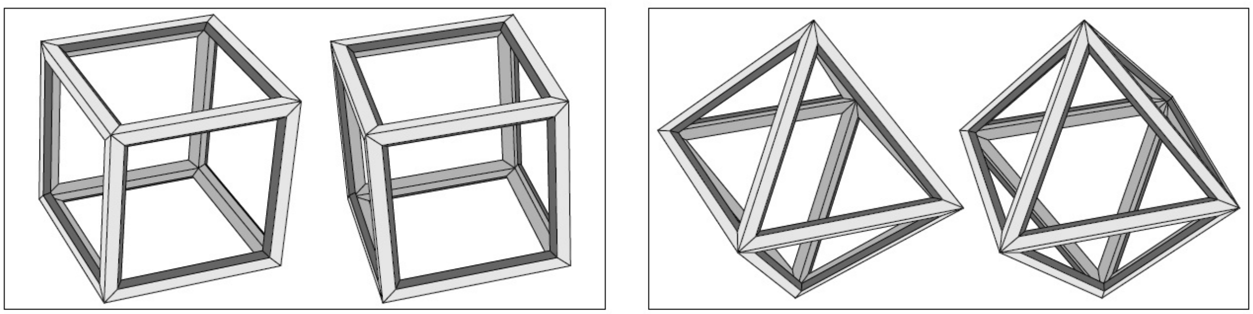


Figure 2.25. Cubes and octahedrons

An *icosahedron* is a 20-faced solid where each face is an equilateral triangle, as shown in Figure 2.26 left. A *dodecahedron* is a 12-faced solid where each face is a regular pentagon, as shown in Figure 2.26 right.

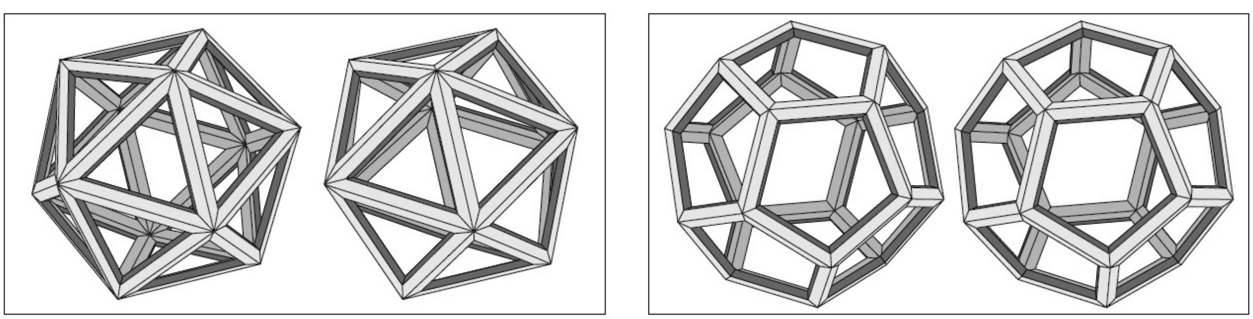


Figure 2.26. Icosahedron and dodecahedron

Note. Plato (circa 425 BCE–circa 348 BCE) in his dialogue *Timaeus*. The nature of the universe is the topic of discussion between Timaeus, Socrates, Hermocrates, and Critias. Timaeus argues that there are four elements, earth, air, fire, and water. He associates the elements with the Platonic solids as:

Element	fire	earth	air	water	ether
Solid	tetrahedron	cube	octahedron	icosahedron	dodecahedron

The dodecahedron is associated with the "ether" or "aether" is the fifth element which makes up the heavens. This idea is mentioned as late as the 1600's when Johannes Kepler (December 27, 1571–November 15, 1630) includes a reference to this in an illustration in his *Harmonices mundi* [Harmonies of the Worlds] (1619); see Figure 2.27.

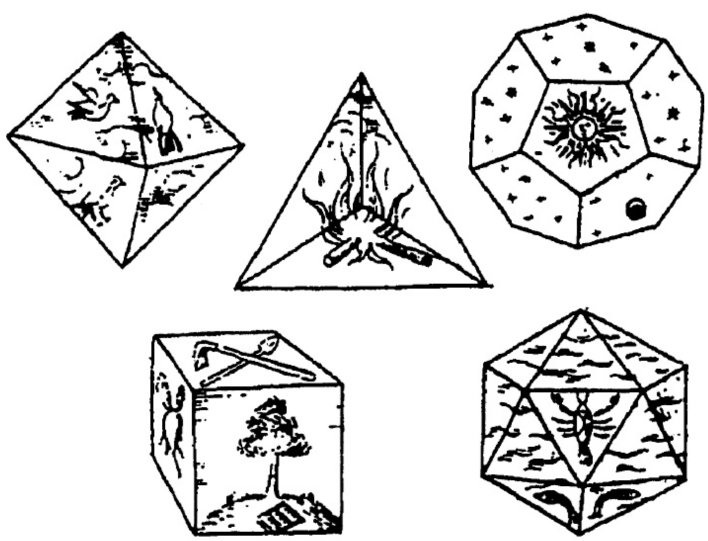
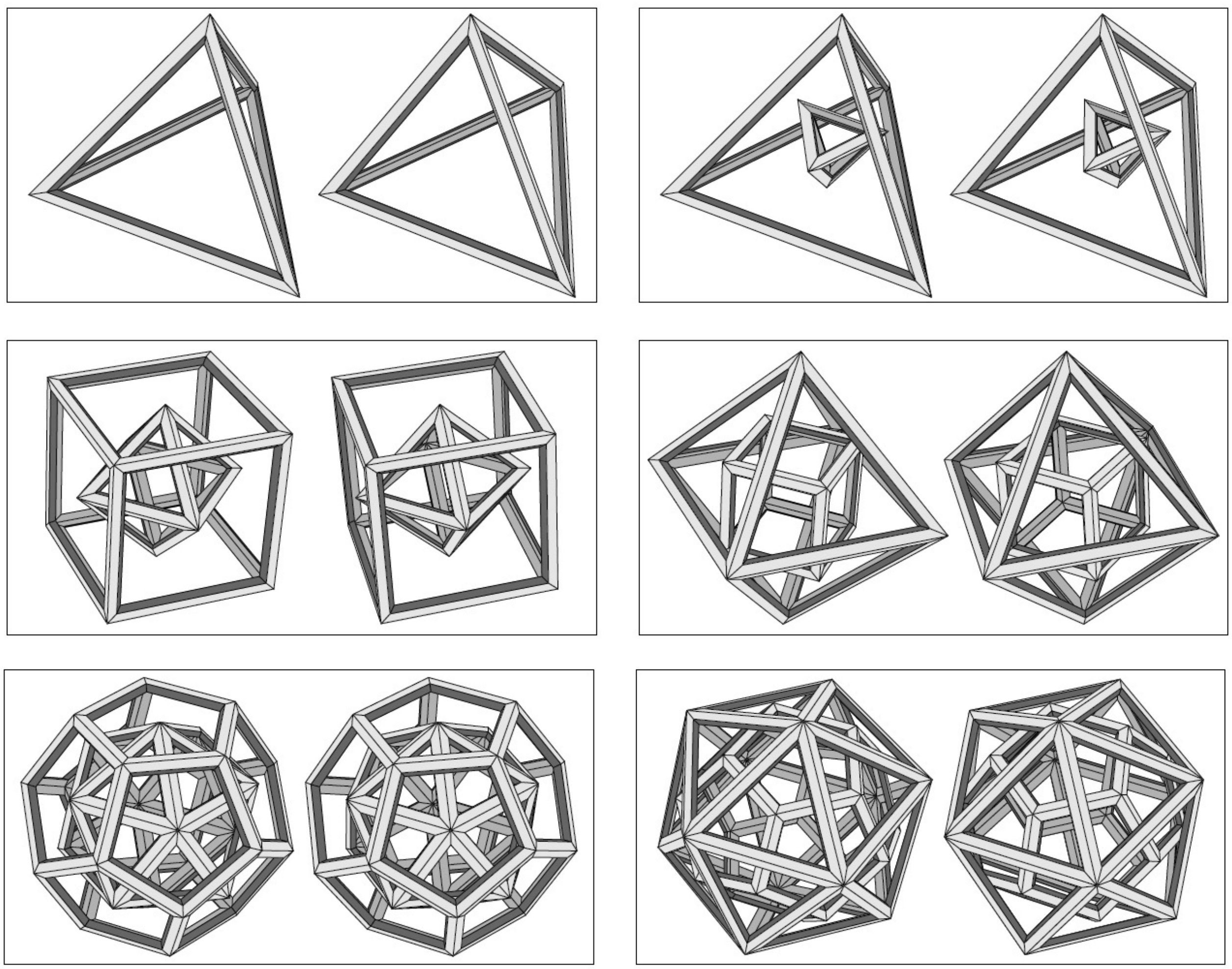


Figure 2.27. Platonic solids and associated elements, from Kepler's *Harmonices mundi*, page 79 (1619)

We will explore some of Kepler's attempts to use the Platonic solids in the explanation of the movements of the planets in our Section 5.10. The Great Discoveries of Kepler and Newton.

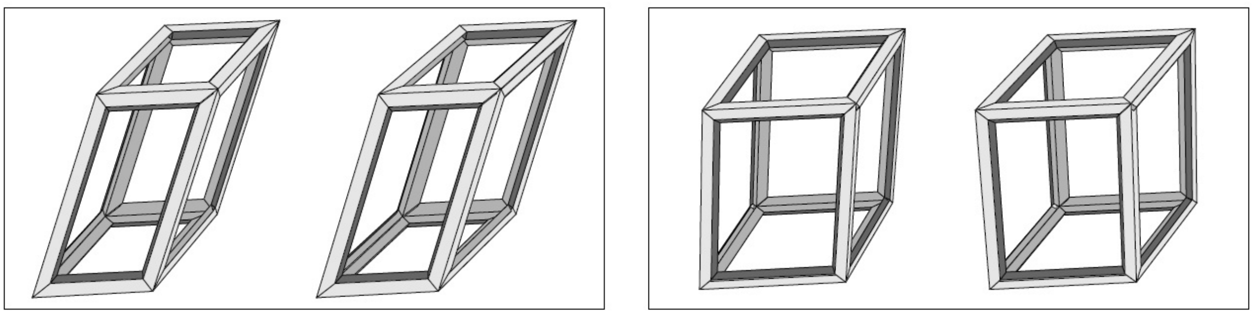
Note/Definition. A geometric property of the Platonic solids is that of *duality*. The dual of a Platonic solid is created by putting a point in the center of each face and the joining the points which are on faces that share an edge. This results in a new solid called the *dual* of the original solid. The dual of a tetrahedron is a tetrahedron (the tetrahedron is said to be *self-dual*); see Figure 2.28. The dual of a cube is a tetrahedron, and vice versa; see Figure 2.29. The dual of the icosahedron is the dodecahedron, and vice versa; see Figure 2.30.



Figures 2.28 to 2.30. Illustrations of the duals of the Platonic solids

We can associate a graph with each Platonic solid and the idea of duality can then be addressed in that setting. See my online notes for Graph Theory 2 (MATH 5450) on Section 10.2. Duality. This idea can be used to show that there are only five Platonic solids; see my online notes for Foundations and Structure of Mathematics 1 (MATH 5025) on The Platonic Solids where Euler's formula is used to establish this claim.

Note. A *parallelepiped* is a solid with parallel surfaces and a *right-angled parallelepiped* is a parallel-piped where all angles are right angles. See Figure 2.31.



Figures 2.31. Parallelepiped and right-angled parallelepiped

You may have encountered the idea of a parallelepiped in Calculus 3 (MATH 2110) in the setting of the scalar triple product; see my online notes for Calculus 3 on Section 12.4. The Cross Product . In Book XI, Euclid presents several results on parallelepipeds concerning angles, volumes, and similarity.

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