Section 2.6. Book XII. Areas and Volumes of Circles, Pyramids, Cones

Note. In this section we complete our study of Euclid's *Elements of Geometry* and briefly consider the content of Books XII and XIII. In particular, we consider the Platonic solids and explain some details not explicitly mentioned in Euclid's study of the Platonic solids at the end of the *Elements*.

Note. Book XII has no definitions and 18 propositions. The first two propositions concern the area of a circle. In particular, Euclid states:

Euclid, Book XII Proposition 1. Similar polygons inscribed in circles are to one another as the squares on their diameters.

This is Euclid's first step in the direction of expressing the area of a circle as proportional to the square of the radius ("$A = \pi r^2$"). Euclid XII.2 completes Euclid's consideration of of the area of a circle (rather unceremoniously, as is his style).

Euclid, Book XII Proposition 2. The areas $\mathcal{A}_1$ and $\mathcal{A}_2$ of two circles C_1 and C_2 of radii r_1 and r_2, respectively, satisfy: $r_2/r_1 = q \Rightarrow \mathcal{A}_1/\mathcal{A}_2 = q^2$.

Note. Since Euclid XIII.2 implies $\mathcal{A}_1/\mathcal{A}_2 = q^2 = (r_2/r_1)^2$, then we have that the area of a circle is proportional to its radius square. That is, $\mathcal{A} = kr^2$ for some constant k; of course, the constant is $k = \pi$. Euclid does not deal with the value of the constant. The famous classical approximation is due to Archimedes of Syracuse (287 BCE–212 BCE), who gave the bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$ in his *Measurement of a Circle*. This is explored in Exercise 2.22 in Ostermann and Wanner and in more detail in my online PowerPoint presentation on Archimedes: 2000 Years Ahead of His Time. The remaining 16 propositions of Book XII concern volumes of pyramids and cones.

Note. Propositions 3 through 9 of Book XII deal with volumes of pyramids. A general result concerning the volumes of pyramids is: $\mathcal{V} = \mathcal{A}h/3$ where $\mathcal{A}$ is the area of the base and h is the altitude of the pyramid.

Note. Propositions 10 through 15 of Book XII deal with volumes of cylinders and cones. In this direction we have the familiar $\mathcal{V}_{\text{cylinder}} = \pi r^3 h$ and $\mathcal{V}_{\text{cone}} = \pi r^2 h/3$. Similar to Euclid XII.2 for circles, Euclid state the following for spheres:

Euclid, Book XII Proposition 17. The volumes $\mathcal{V}_1$ and $\mathcal{V}_2$ of two spheres with radius r_1 and r_2, respectively, satisfy $r_2/r_1 = q \Rightarrow \mathcal{V}_2/\mathcal{V}_1 = q^3$.

Similar to the observation about Euclid XII.2, we have from Euclid XII.17 that $\mathcal{V} = 4\pi r^3/3$ (also a result of Archimedes, this time from his *On Conoids and Spheroids*). Ostermann and Wanner (page 51) observe the "beautiful relation" of ratios (which requires us to treat the altitude of a sphere as $h = 2r$ so that $\mathcal{V}_{\text{sphere}} = 4\pi r^3/3 = 2\pi r^2 h/3$):

$$\mathcal{V}_{\text{cone}} : \mathcal{V}_{\text{sphere}} : \mathcal{V}_{\text{cylinder}} = \pi r^2 h/3 : 2\pi r^2 h/3 : \pi r^2 h = 1 : 2 : 3,$$

where we can take the cylinder as circumscribing the sphere.

Note. Book XIII has no definitions and 18 propositions. The first dozen or so propositions deal largely with inscribing regular polygons in circles. Propositions 13 through 17 give the constructions of a tetrahedron, octahedron, cube, icosahedron, and dodecahedron, respectively. The constructions are based on embedding the solids in a sphere. Euclid XIII.13 involves the construction of a pyramid, but since a tetrahedron is a pyramid with an equilateral triangle as its base, then this covers the construction of a tetrahedron). In Euclid XIII.17, the construction starts with a cube and the "hipped roofs" are added to each face; see Figure 2.37. In a final remark in the *Elements*, Euclid states: "No other figure, besides the said five figures, can be constructed by equilateral and equiangular figures equal to one another."

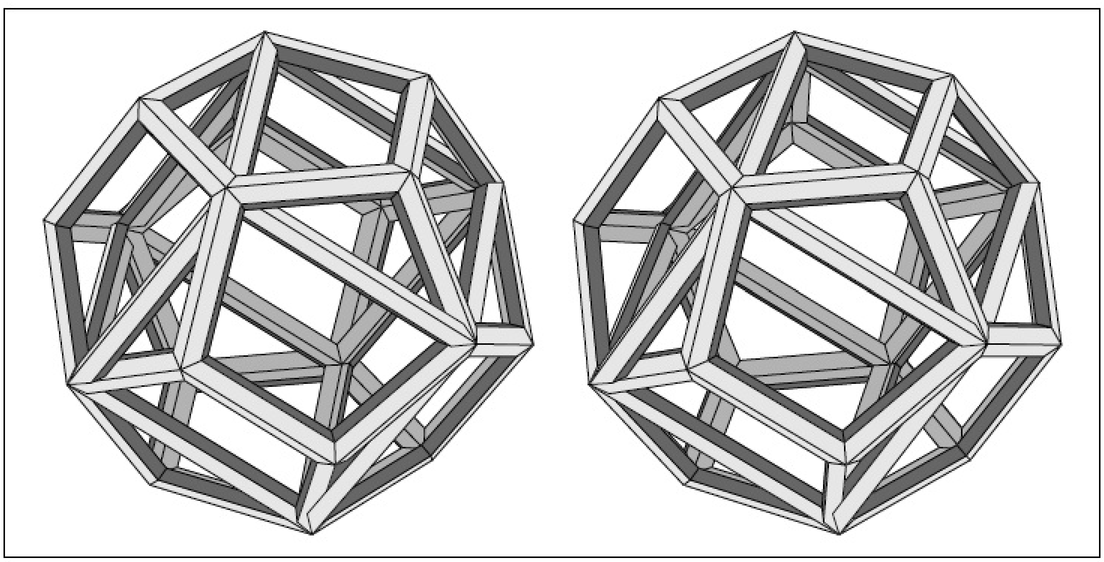
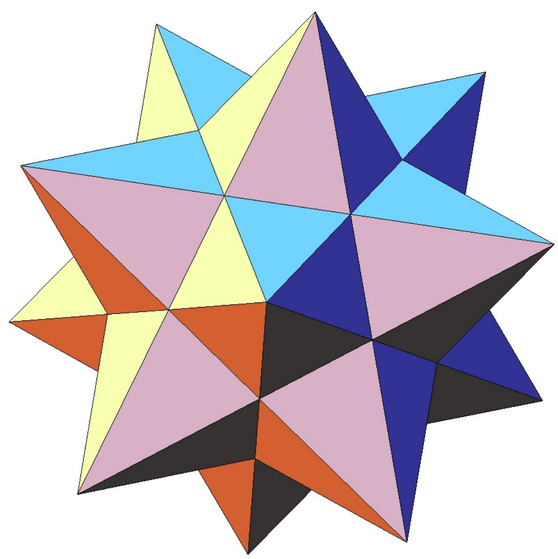


Figure 2.37. The dodecahedron built on a cube

Note. Benno Artmann in *Euclid—The Creation of Mathematics* (NY: Springer-Verlag, 1999) states (see pages 296–297):

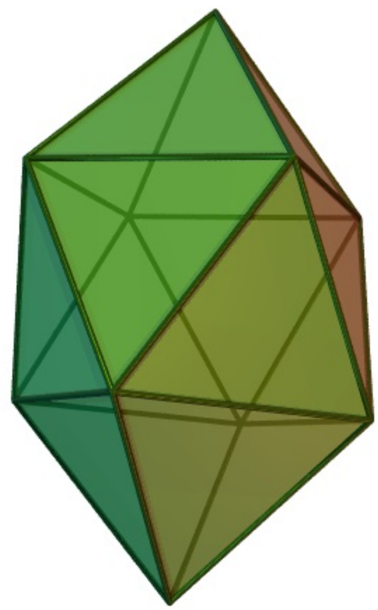
> "Euclid's treatment of the regular polyhedra is especially important for the history of mathematics because it contains the first example of a major classification theorem.... Euclid apparently regards a solid as regular if its faces are congruent regular polygons. This is essentially the modern definition except for two omissions: the requirement that the polyhedron be convex, and the specification that the solid angles (vertex figures) of the polyhedron be congruent."

Euclid likely simply assumed convexity in his presentation. To show that it is necessary, consider the stellated dodecahedron which has all faces as equilateral triangles:



From the Wikipedia "List of Wenninger polyhedron models" page

Notice that at some vertices there are 5 triangles and at others there are 6 triangles. To see that the congruence of the solid angles of the polyhedron is necessary, consider the gyroelongated square bipyramid:



From the Wikipedia "Gyroelongated square bipyramid" page

This has 16 equilateral triangles as its faces. It is convex, but the solid angles are not congruent since there are 4 triangles at some vertices and 5 triangles at other vertices.

Note. Since the construction of the Platonic solids comes at the very end of the *Elements*, it has been speculated that the purpose of the *Elements* is to reach this conclusion, what with its cosmic significance and all, as discussed in Section 2.5. Book XI. Spatial Geometry and Solids. Artmann in his book mentioned above states (see page 298):

> "Proclus states that the aim of the Elements is 'both to furnish the learner with an introduction to the science as a whole and to present the construction of the several cosmic figures (the regular solids)' [Proclus-Morrow, page 59] This characterization is too humble. The Elements contains much more than a first introduction to mathematics, and, as we have seen, the study of the regular solids is only one of several highlights." (page 298)

(The Proclus reference is mentioned in our introduction to Chapter 2. The Elements of Euclid.) Euclid's *Elements of Geometry* contains way more material than a simple build up to the classification of Platonic solids!

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