Section 2.7. Epilogue

Note. In section Section 2.1. Book I, we expressed some concerns about Euclid's approach to geometry in the *Elements*. In particular, we expressed concern over the useless nature of some of his definitions, his absence of consideration of continuity (see Euclid I.1), and his informal use of superposition or "coincidence" (see Euclid I.4). For 2000 years, attempts were made to modify Euclid's axioms/postulates to remedy these concerns and to give a proof of the Parallel Postulate in terms of the other axioms/postulates.

Note. We explored the attempts to proof the Parallel Postulate in Section 2.1. Book I with the work of Saccheri, and the ultimate results of Gauss, Lobachevsky, and Bolyai on hyperbolic geometry in the early 1800s.

Note. An influential result is Adrian-Marie Legendre's (September 1752–January 10, 1833) 1794 *Elements of Geometry*. In the Britannica Legendre biography, it is stated:

> [In 1794] he published *Éléments de géométrie* (*Elements of Geometry*), a reorganization and simplification of the propositions from Euclid's *Elements* that was widely adopted in Europe, even though it is full of fallacious attempts to defend the parallel postulate. Legendre also gave a simple proof that π is irrational, as well as the first proof that π^2 is irrational, and he conjectured that π is not the root of any algebraic

equation of finite degree with rational coefficients (i.e., π is a transcendental number). His *Éléments* was even more pedagogically influential in the United States, undergoing numerous translations starting in 1819; one such translation went through some 33 editions.

Note. David Hilbert (January 23, 1862–February 14, 1943) published *Grundlagen der Geometrie* (*The Foundations of Geometry*) in 1899. He presented 21 axioms (called today "Hilbert's axioms of geometry") for a modern axiomatic systems of Euclidean geometry. In 1902, E. H. Moore and (independently) R. L. Moore proved that one of Hilbert's axioms could be proved from the other axioms, so that Hilbert ultimately has 20 axioms. He has three undefined terms (point, line, and plane) and three undefined relations (betweenness, lies on, and congruence). The axioms (and six "primitive notions") then give properties of the undefined terms and undefined relations. You can find an online copy of Hilbert's work in English at the Project Gutenberg copy of *The Foundations of Geometry* (translation of E. J. Townsend, Open Court Publishing, 1950); this version has only 87 numbered pages (accessed 1/28/2022).

David Hilbert (January 23, 1862–February 14, 1943)

From the MacTutor History of Mathematics Archive Hilbert biography page

Note. In 1932, George Birkhoff (March 21, 1884–November 12, 1944) gave four axioms (called "Birkhoff's axioms") for Euclidean geometry, but worked under the assumption that the real numbers are given (or defined; the real numbers are a complete, ordered field). He used the real numbers to set a scale and protractor for the Euclidean plane. His work appears as "A Set of Postulates for Plane Geometry (Based on Scale and Protractors)," *Annals of Mathematics*, **33**(2), 329-345 (1932). You can view Birkhoff's paper on JSTOR (but you will need to sign in with your ETSU username and password; accessed 1/29/2022).

Note. An axiomatic approach to Euclidean geometry is given in my online notes for Introduction to Modern Geometry (MATH 4157/5157) on The Axiomatic Method. The approach given in these notes is similar to that of Birkhoff and sets up a scale (see Postulate 11, The Ruler Postulate, in Section 2.4. The Measurement of Distance) and a protractor (see Postulate 15, The Protractor Postulate, in Section 2.6. Angles and Angle Measurement). The approach to geometry in these Introduction to Modern Geometry notes has 21 axioms/postulates; the emphasis is on rigor and understanding over minimization of axioms.

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