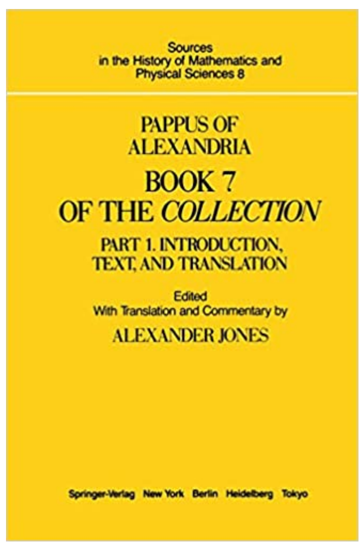
Section 3.1. The Parabola.

Note. As described in the introduction to Chapter 3, Menaechmus introduced a parabola as an intersections of a plane with cone. We'll see in Figure 3.2 (center) below that the plane must be parallel to some line on the cone which passes through the vertex of the cone (such a line is called a *generator* of the cone). The result we give (which is stated as a definition) is due to Pappus of Alexandria (circa 290–circa 350) and appeared in his *Mathematical Collection*, Book VII, Proposition 238. Though I cannot find an image of Pappus online (the images in these notes are all referenced, but their historical accuracy is questionable). However, some of his work is still in print, including Book VII.



From Amazon.com (accessed 9/21/2021)

Definition. Let d be a line, called the *directrix*, and F a point, called the *focus*, at distance p from the directrix. The locus of all points P that have the same given distance ℓ from F as from f is called a *parabola*. See Figure 3.1 (left).

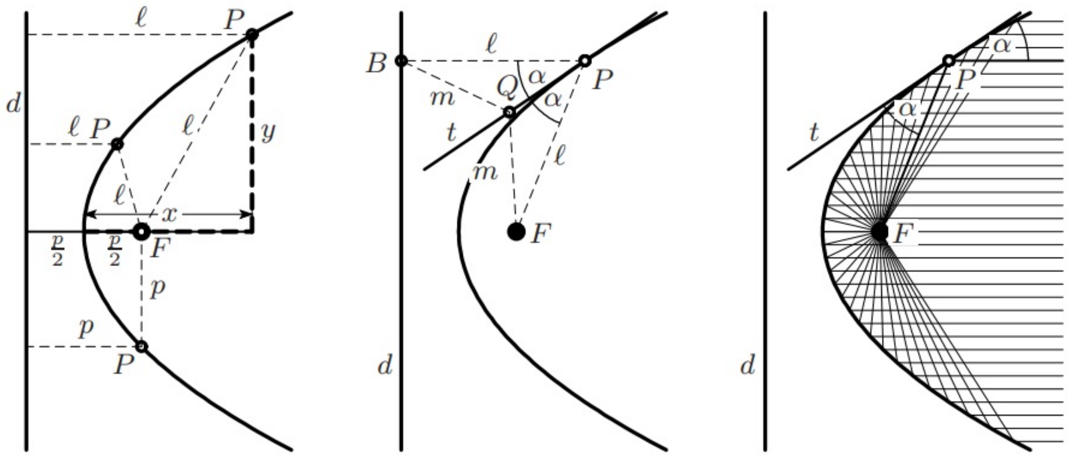


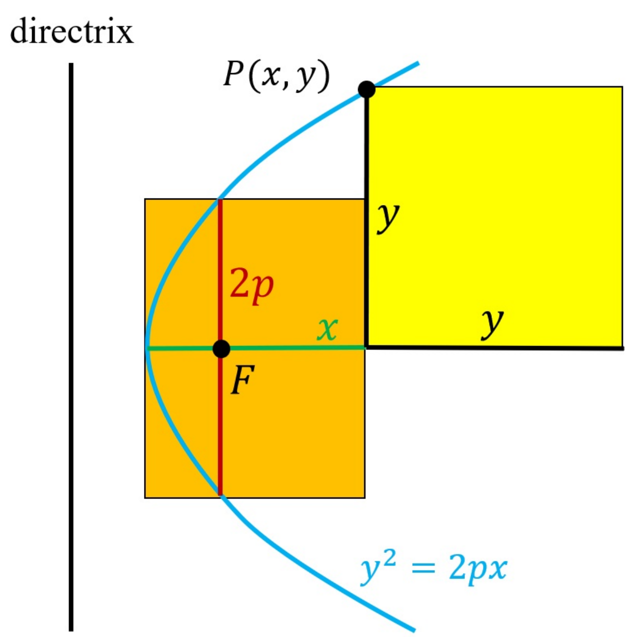
Fig. 3.1. Definition and tangent of a parabola

Note 3.1.A. The definition above is the same one used, for example, in Calculus 3 (MATH 2110). See my online notes for this class on Section 11.6. Conic Sections. A parabola is symmetric with respect to a line perpendicular to the directrix which passes through the focus. This line is called the *axis* of the parabola and the point at which it intersects the parabola is the *vertex* of the parabola. We can derive an equation for a parabola using Figure 3.1 (left). Let point $P = (x, y)$ be on the parabola in the coordinate system with its origin at the vertex and with the x-axis horizontal. Notice for point P in the figure, the distance ℓ from P to F is the same as the distance from P to the directrix and this distance is $\ell = x + p/2$. Now in the right triangle with vertices P and F and hypotenuse given by the line segment PF, the Pythagorean Theorem gives that

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \ell^2 = \left(x + \frac{p}{2}\right)^2 \text{ or } x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4} \text{ or } y^2 = 2px.$$

The value $2p$ is the length of a vertical line segment through the focus from one side of the parabola to the other (in Figure 3.1, left); this is called the *latus rectum* and value p is the *semi latus rectum*.

Note 3.1.B. As the following figure shows, any point $P(x, y)$ on the parabola $y^2 = 2px$ determines a rectangle of width x and height $2p$, and a square of width y and height y. Since the rectangle has area $2px$ and the square has area y^2, then these areas are the same since x and y are related by the equation $y^2 = 2px$. As stated on page 62 of Ostermann and Wanner, according to Henry Liddell and Robert Scott's *Greek-English Lexicon* (Oxford Press, 2016), "parabola" (in Greek, $\pi\alpha\rho\alpha\beta o\lambda\acute{\eta}$) means comparison, juxtaposition, or analogy. The equality of the areas of the rectangle and square motivated Apollonius of Perga (circa 262 BCE–circa 190 BCE) to name the curve a *parabola*.



Note. Paraphrasing Apollonius himself, from Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections*, Edited in Modern Notation (Cambridge University Press, 1896), we have the following (where we have changed his labels of points so that they agree with the above picture; see Apollonius' page 9):

> It follows that the square on any ordinate y to the axis of the parabola [Apollonius calls the axis the "fixed diameter"] is equal to a rectangle applied ($\pi\alpha\rho\alpha\beta\acute{\alpha}\lambda\lambda\epsilon\iota\nu$) to the fixed straight line of length $2p$ [the latus rectum] to the fixed straight line drawn at right angles to the axis of the parabola, with altitude equal to the corresponding abscissa x. Hence the section is called a PARABOLA.

Note. In the next result, the intersection of a plane with a cone of Menaechmus is shown to give the same curve as the one given by Pappus' definition. The proof we present is due to Germinal Dandelin (April 12, 1794–February 15, 1847) and presented in his "Memoir on some remarkable properties of the parabolic focale [i.e., oblique strophoid]," *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Bruxelles* (in French), **2**, 171-200 (1822). The technique used employs so-called "Dandelin spheres." Credit for this approach is also sometimes given to Adolphe Quetelet based on his "Dissertatio mathematica inauguralis de quibusdam locis geometricis nec non de curva focali [Inaugural mathematical dissertation on some geometric loci and also focal curves]," doctoral thesis (University of Ghent, Belgium, 1819). This is also the proof technique used in these notes to show that an ellipse and hyperbola also result from conic sections, as is shown in Theorem 3.2.A and Theorem 3.2.B.

Theorem 3.1. (Apollonius' Proposition I.11 in *Treatise on Conic Sections*)
If a cone is cut by a plane that has the same slope as the generators of the cone, then the intersection is a parabola.

Note 3.1.C. We now consider lines tangent to a parabola. This is straightforward in calculus using derivatives. But here we give an argument based on Euclidean geometry and a construction of the tangent line that could be performed with a compass and straightedge.

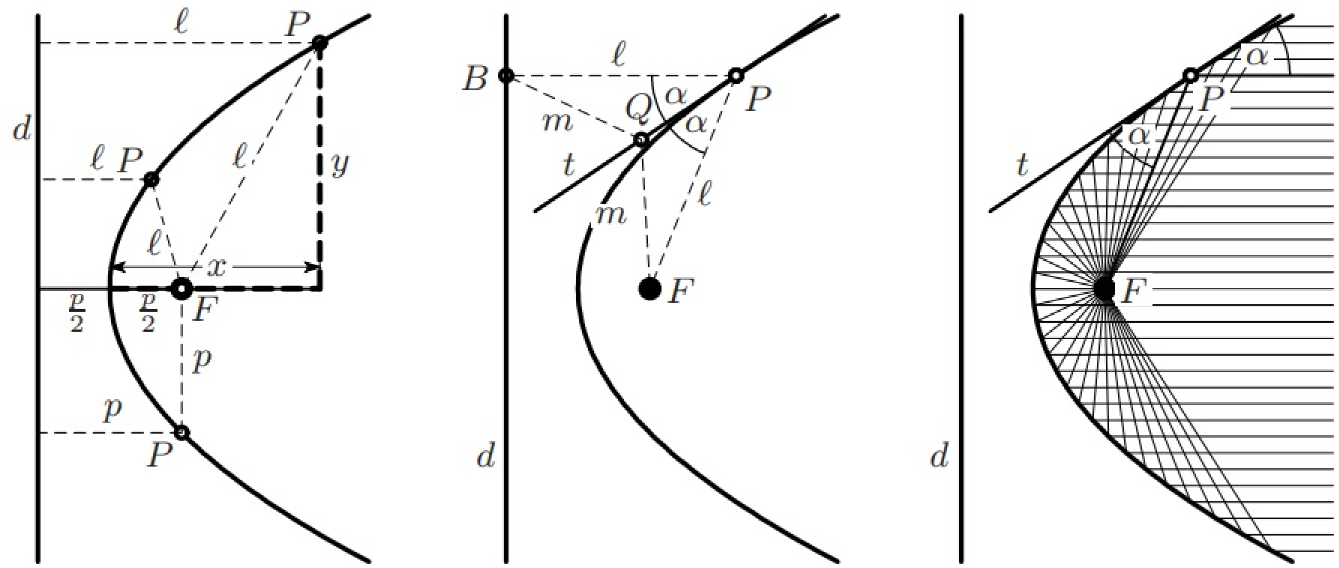


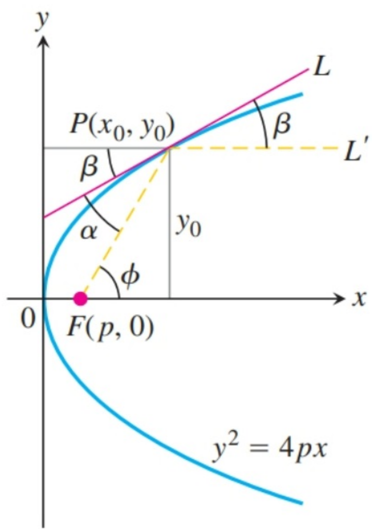
Fig. 3.1. Definition and tangent of a parabola

Let P be an arbitrary point on the parabola and let line t be the bisector of the angle BPF, as given in Figure 3.1 (center). For Q another point on line t, we have the lengths of segments BQ and QF are the same since triangles BPQ and FPQ are congruent (by Side-Angle-Side, say). Now segment QF is longer that the distance from Q to the directrix line d, since BQ (which is of the same length at QF) is not orthogonal to d (and so is longer that the distance from Q to the directrix). So all points of line t (other than point P) lie outside the parabola. Therefore, line t is tangent to the parabola at point P.

Note. Euclid I.15 states: "If two straight lines cut one another, then they make the vertical angles equal to one another." So if we extend the line segment BP in Figure 3.1 (center), then we can use it to represent a ray of light approaching the parabola from the right. The Law of Reflection states that the angle of incidence equals the angle of reflection. So if the parabola is a reflective surface, then when a ray of light comes in from the right parallel to the axis then its angle of incidence will be α and so its angle of reflection will also be α (see Figure 3.1, right). Then, as just argued, the ray of light will travel to the focus (following the path from point P to point F). This can be shown using calculus as well, as we show next.

Note 3.1.D. In Calculus 3 (MATH 2110), the reflective property of a parabola is addressed in Section 11.6. Conic Sections. In Exercise 81 from this section of *Thomas' Calculus, Early Transcendentals*, 12th Edition (see page 666), we have the following. Notice that Ostermann and Wanner take the distance between the vertex and focus as $p/2$, whereas *Thomas' Calculus* takes this distance to be p. So our equation $y^2 = 2px$ is equivalent to their equation $y^2 = 4px$.

Exercise 11.6.81. The accompanying figure shows a typical point $P(x_0, y_0)$ on the parabola $y^2 = 4px$. The line L is tangent to the parabola at P. The parabola's focus lies at $F(p, 0)$. The ray L' extending from P to the right is parallel to the x-axis. We show that light moving to the left along L' reflects off the parabola at point P to then travel to the focus F by showing that β equals α.



Solution. First, the slope of line L equals that tangent of β. Since $y^2 = 4px$ then, by implicit differentiation, $2y\,dy/dx = 4p$ or $dy/dx = 2p/y$. So at point $P(x_0, y_0)$ the slope of a tangent line to the parabola is $(dy/dx)|_{(x,y)=(x_0,y_0)} = 2p/y_0$, and hence $\tan\beta = 2p/y_0$. Similarly, $\tan\phi$ is the slope of the line through points $F(p,0)$ and $P(x_0, y_0)$. That is, $\tan\phi = (y_0 - 0)/(x_0 - p) = y_0/(x_0 - p)$. Since $\alpha + \beta + (\pi/2 - \phi) = \pi/2$, then we have $\alpha = \phi - \beta$. By the trigonometric identity $\tan(\theta_0 - \theta_1) = \dfrac{\tan\theta_0 - \tan\theta_1}{1 + \tan\theta_0 \tan\theta_1}$ we now have

$$\begin{aligned}
\tan\alpha &= \tan(\phi - \beta) = \frac{\tan\phi - \tan\beta}{1 + \tan\phi\tan\beta} \\
&= \frac{y_0/(x_0 - p) - 2p/y_0}{1 + (y_0/(x_0 - p))(2p/y_0)} = \frac{y_0 - 2p(x_0 - p)/y_0}{(x_0 - p) + 2p} \\
&= \frac{y_0 - 2px_0/y_0 + 2p^2/y_0}{x_0 + p} = \frac{y_0^2 - 2px_0 + 2p^2}{(x_0 + p)y_0} \\
&= \frac{(4px_0) - 2px_0 + 2p^2}{(x_0 + p)y_0} \text{ since } y_0^2 = 4px_0 \\
&= \frac{2px_0 + 2p^2}{(x_0 + p)y_0} = \frac{2p(x_0 + p)}{(x_0 + p)y_0} = \frac{2p}{y_0}.
\end{aligned}$$

Therefore, $\tan\beta = 2p/y_0 = \tan\alpha$ and, since α and β are both acute, we have

$\alpha = \beta$. Since the angle of incidence is β and $\alpha = \beta$, then α is the angle of reflection (by the Law of Reflection) and hence light traveling to the left along L' reflects off of the parabola at point P and goes to the focus F, as claimed. $\square$

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