Section 3.2. The Ellipse.

Note. We again consider the intersection of a plane with a cone. In the previous section, we considered the case where the plane was parallel to some generator of the cone (and this resulted in a parabola). In this section we consider the case where the plane is "less steep" than the generators of the cone (see Figure 3.3, left). Notice that this results in curve of intersection in the plane which is bounded (see Figure 3.3 right for a projection of the curve onto a vertical plane).

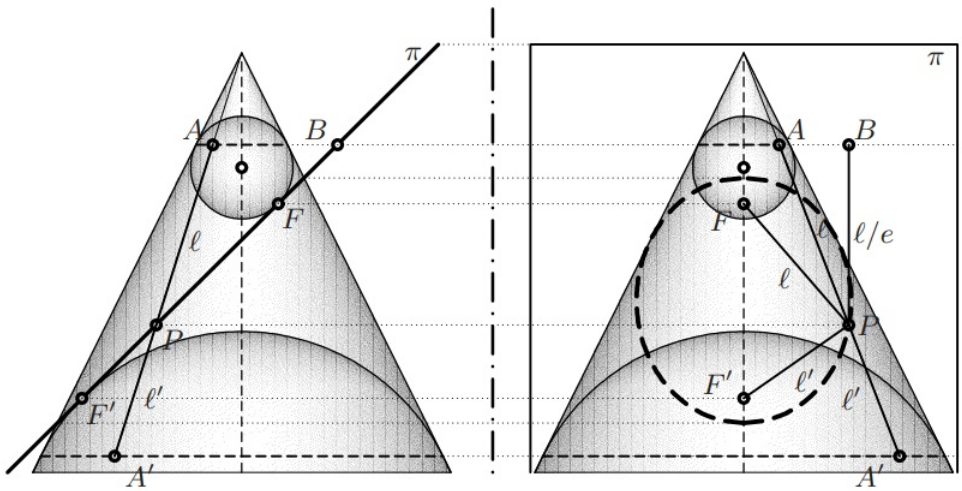


Fig. 3.3. An ellipse as the intersection of a cone with a plane

The next definition is, according to Osermann and Wanner, due to Pappus of Alexandria (circa 290–circa 350). A specific reference is not given, but presumably it appeared in his *Mathematical Collection*, as did the definition of a parabola.

Definition. Let d be a line (called the *directrix*) and F a point (called the *focus*) at distance p/e from the directrix, where $0 < e < 1$ is the *eccentricity*. The locus of all points P for which the ratio of the distances to the point F and to the line d equals e is called an *ellipse*. See Figure 3.4 (left).

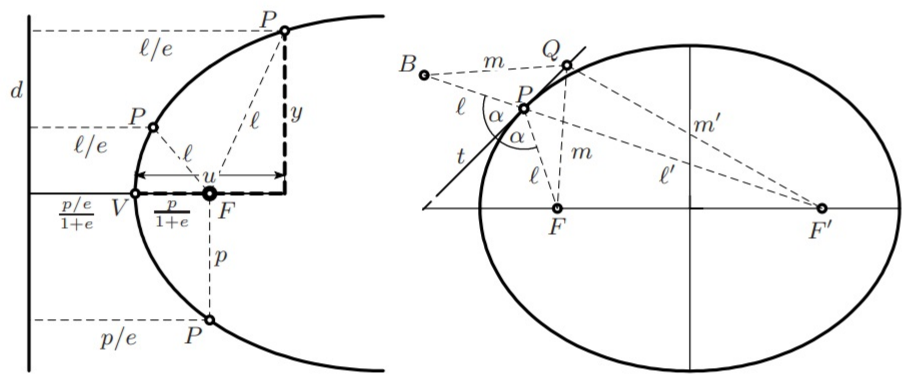


Fig. 3.4. Definition of an ellipse and construction of a tangent

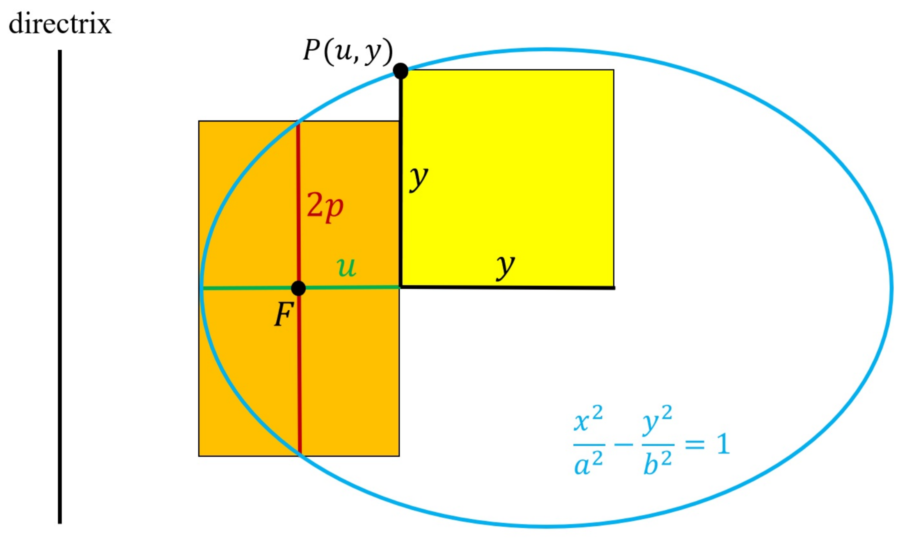
Note. Notice that if we consider $e = 1$, then the above definition reduces to the definition of a parabola. Define the point on the ellipse which is closest to the focus as the *vertex*, which we denote V. Notice that the point V must be a distance of $p/(1+e)$ from focus F, and V must be a distance of $(p/e)/(1+e)$ from the directrix d, so that the ratio of the distance from V to F to the distance from V to d is $\dfrac{p/(1+e)}{(p/e)/(1+e)} = e$, as required. See Figure 3.4 (left). Let point $P = (u, y)$ be on the ellipse in the coordinate system with its origin at the vertex and with the u-axis horizontal. We consider the right triangle with vertices P and F and hypotenuse given by the line segment PF. The legs of the triangle are then of lengths $u - p/(1+e)$ and y, and we let the hypotenuse have length ℓ. Notice that we then have that point P is a distance of ℓ/e from directrix d, and that $\dfrac{\ell}{e} = u + \dfrac{p/e}{1+e}$ and so $\ell = e\left(u + \frac{p/e}{1+e}\right)$. So by the Pythagorean Theorem we have

$$\left(u - \frac{p}{1+e}\right)^2 + y^2 = \ell^2 = e^2\left(u + \frac{p/e}{1+e}\right)^2 = \left(eu + \frac{p}{1+e}\right)^2,$$

$$\text{or } u^2 - \frac{2up}{1+e} + \frac{p^2}{1+e^2} - e^2u^2 - \frac{2eup}{1+e} - \frac{p^2}{1+e^2} + y^2 = 0,$$

$$\text{or } u^2 - \frac{2up(1+e)}{1+e} - e^2u^2 + y^2 = 0 \text{ or } (1-e^2)u^2 - 2up + y^2 = 0. \qquad (3.3)$$

Note 3.2.A. As the following figure shows, any point $P(u, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ determines a rectangle of width u and height $2p$, and a square of width y and height y. Since the rectangle has area $2up$ and the square has area y^2, then the area of the square is less than the area of the rectangle because $y^2 = 2up - (1 - e^2)u^2$ (where $0 < e < 1$) by (3.3). As stated on page 64 of Ostermann and Wanner, according to Henry Liddell and Robert Scott's Greek-English Lexicon (Oxford Press, 2016), "ellipse" (in Greek, ἐλλῐπής) means leaving out, omitting, or lack. The inequality given by the fact that the are of the square is less than the area of the rectangle motivated Apollonius of Perga (circa 262 BCE–circa 190 BCE) to name the curve an *ellipse*.



Note. Paraphrasing Apollonius himself, from Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections*, Edited in Modern Notation (Cambridge University Press, 1896), we have the following (where we have changed his labels of points so that they agree with the above picture; see Apollonius' page 12):

> Thus the square on the ordinate (that is, the square with area y^2) is less than a rectangle whose height is equal to the latus rectum $2p$ and whose base is the abscissa x. [Apollonius actually speaks of the equality of the square and a rectangle whose height falls short of the latus rectum by a certain amount. In this way he is dealing with an equation of the form $y^2 = px - (p/d)x^2$.] The section is therefore called an ELLIPSE.

Note. The next result shows that the intersection of a plane (of certain steepness) with a cone produces an ellipse. This allows us to connect Menaechmus' approach to ellipses with the idea of a locus of points whose distances from two fixed points (foci) have a constant sum. We discuss below how these ideas are combined with the directrix and focus approach given above. The result appears in Apollonius' *Treatise on Conics* as Proposition III.52. The proof uses Dandelin spheres, as did Theorem 3.1 and as will Theorem 3.3.A. In fact, the next result appears as Exercise 5.10 of Keith Kendig's *Conics*, The Dolciani Mathematical Expositions #29, Mathematical Association of America (2005).

Theorem 3.2.A. (Apollonius' Proposition III.52 in *Treatise on Conic Sections*) The intersection of a cone and a plane that is less steep than the generators of the cone is a locus of all points in a plane whose distances from two fixed points in the plane (called *foci*) have a constant sum.

Note. The parameter e in the proof of Theorem 3.2.A is the eccentricity of the ellipse. The line of intersection of plane π and the plane containing circle C is the directrix of the ellipse. Instead of Pappus' definition of an ellipse in terms of a focus and a directrix, we can take the following as an alternate definition.

Definition. A *ellipse* is the locus of all points P in a plane whose distances from two fixed points in the plane have a constant sum.

Note 3.2.B. In fact, it is this second definition that we use in Calculus 3 (MATH 2110; see my online notes for Calculus 3 on Section 11.6. Conic Sections). It is shown there that the formula for an ellipse is equivalent to the formula given in (3.3), though some translation may be necessary to get the formulas to agree. This result, combined with Theorem 3.2.A, shows that the two definitions of an ellipse are equivalent. Since the proof of Theorem 3.2.A shows that the intersection of a plane with a cone (where the plane is "less steep" than the generators of the cone) is a locus of points whose distances from two points in the plane is a constant sum, then we have that such an intersection (or "conic section") is a locus of points that satisfies the two definition of an ellipse given above. That is, we have unified the following three types of definitions of an ellipse:

1. as the intersection of a cone and a "less steep" plane (Menaechmus' definition),
2. in terms of two fixed points, foci, and the sum of distances being constant (the second definition; Apollonius' Proposition III.52), and
3. in terms of a directrix and focus (Pappus' definition).

Note. We now consider compass and straight edge constructions of tangents to an ellipse, as we did for a parabola in Note 3.1.C. The technique is stated in Apollonius' *Treatise on Conic Sections*:

Theorem III.48. (Apollonius, *Treatise of Conics*)
For an ellipse (and a hyperbola), the focal distances of P make equal angles with the tangent at that point.

The argument for this is based on Figure 3.4, right (below). Let point P on the ellipse. Join P to F and F' with segments of lengths ℓ and ℓ', respectively. Extend $F'P$ by the distance ℓ to a point B. Bisect angle $\angle BPF$ with line t. For any other point Q on t, the triangle $\triangle BQF$ is isosceles, since triangles $\triangle BQP$ and $\triangle FQP$ are congruent (by SAS, since they share side QP, both sides BP and FP are length ℓ, and angles $\angle BPQ$ and $\angle FPQ$ are equal by the bisection). Therefore BQ and QF and the same length, say m. With m' as the length of $F'Q$, we have $m + m'$ is longer than $F'B$, since points B, Q, and F' are not collinear (By Euclid Proposition I.20, the sum of any two sides of a triangle is greater than the remaining side; consider triangle $\triangle BQF'$). Therefore all points on line T, other than point P, lie outside the ellipse so that t is tangent to the ellipse at point P.

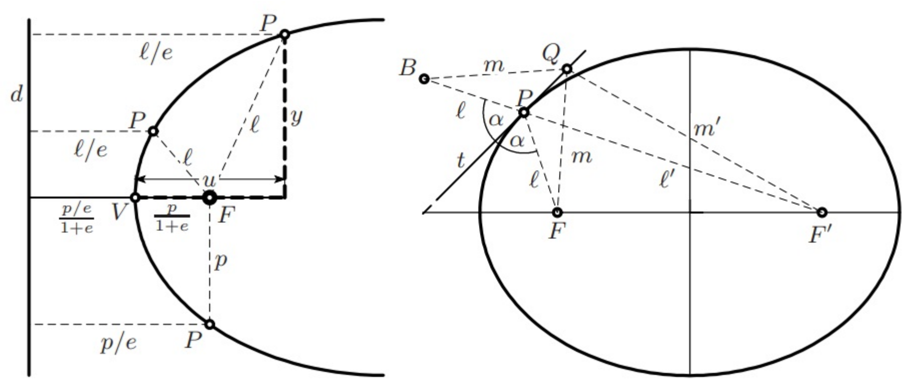
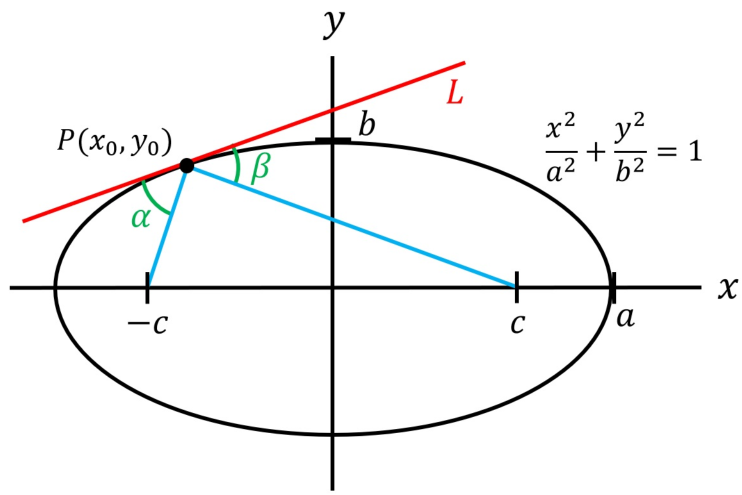


Fig. 3.4. Definition of an ellipse and construction of a tangent

Note 3.2.C. The acute angle θ between two nonperpendicular, nonvertical intersecting lines of slopes m_1 and m_2 satisfies the equation $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$. This follows from the summation formula for tangent, where the absolute value is needed since the angle is required to be acute (the angle "between" two lines does not have an orientation, but if we interchange m_1 and m_2 then $\tan\theta$ changes by a negative sign).



Denote the foci as $F_1(-c, 0)$ and $F_2(c, 0)$ where $c > 0$. Differentiating (implicitly) the formula for the ellipse gives $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = \dfrac{-2x/a^2}{2y/b^2} = \dfrac{-xb^2}{ya^2}$. So the slope of line L in in the above figure is $m_L = \left.\dfrac{dy}{dx}\right|_{(x,y)=(x_0,y_0)} = \dfrac{-x_0 b^2}{y_0 a^2}$. The slope of $\overleftrightarrow{F_1P}$ is $m_{\overleftrightarrow{F_1P}} = \dfrac{y_0 - 0}{x_0 - (-c)} = \dfrac{y_0}{x_0 + c}$. The slope of $\overleftrightarrow{F_2P}$ is $m_{\overleftrightarrow{F_2P}} = \dfrac{y_0 - 0}{x_0 - c} = \dfrac{y_0}{x_0 - c}$. We now have $\tan\alpha$ as

$$\begin{aligned}
\tan\alpha &= \left|\frac{m_L - m_{\overleftrightarrow{F_1P}}}{1 + m_L m_{\overleftrightarrow{F_1P}}}\right| = \left|\frac{\frac{-x_0 b^2}{y_0 a^2} - \frac{y_0}{x_0+c}}{1 + \frac{-x_0 b^2}{y_0 a^2}\frac{y_0}{x_0+c}}\right| \\
&= \left|\frac{x_0 b^2(x_0 + c) + y_0(y_0 a^2)}{y_0 a^2(x_0 + c) - x_0 b^2 y_0}\right| = \left|\frac{x_0 b^2 + x_0 b^2 c + y_0^2 a^2}{x_0 y_0 a^2 + y_0 a^2 c - x_0 y_0 b^2}\right| \\
&= \left|\frac{a^2 b^2 + x_0 b^2 c}{x_0 y_0 (a^2 - b^2) + y_0 a^2 c}\right| \text{ since } \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \text{ so that } x_0^2 b^2 + y_0^2 a^2 = a^2 b^2 \\
&= \left|\frac{b^2(a^2 + x_0 c)}{y_0 c(x_0 c + a^2)}\right| = \left|\frac{b^2}{y_0 c}\right| \text{ since } c^2 = a^2 - b^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tan\beta &= \left|\frac{m_L - m_{\overleftrightarrow{F_2P}}}{1 + m_L m_{\overleftrightarrow{F_2P}}}\right| = \left|\frac{\frac{-x_0b^2}{y_0a^2} - \frac{y_0}{x_0-c}}{1 + \frac{-x_0b^2}{y_0a^2}\frac{y_0}{x_0-c}}\right| \\
&= \left|\frac{x_0b^2(x_0-c) + y_0(y_0a^2)}{y_0a^2(x_0-c) - x_0b^2y_0}\right| = \left|\frac{x_0b^2 - x_0b^2c + y_0^2a^2}{x_0y_0a^2 - y_0a^2c - x_0y_0b^2}\right| \\
&= \left|\frac{a^2b^2 - x_0b^2c}{x_0y_0(a^2-b^2) - y_0a^2c}\right| \text{ since } \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1 \text{ so that } x_0^2b^2 + y_0^2a^2 = a^2b^2 \\
&= \left|\frac{b^2(a^2 - x_0c)}{y_0c(x_0c - a^2)}\right| = \left|\frac{-b^2}{y_0c}\right| \text{ since } c^2 = a^2 - b^2.
\end{aligned}$$

Therefore $\tan\alpha = \tan\beta$ and, since α and β are acute, then $\alpha = \beta$. If a ray of light is emitted at focus F_1 and reflects off of the ellipse at point P, the the angle of incidence is α and since $\alpha = \beta$ then the angle of reflection is β (by the Law of Reflection) and hence the light will travel to the other focus at F_2. This is the reflective property of the ellipse.

Note. As we see in Calculus 3 (in Section 11.6. Conic Sections)) and used in the previous note, an ellipse with foci at $(-c, 0)$ and $(c, 0)$ in Cartesian coordinates has a formula of the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ where for $a \geq b$ we have the sum of distances as $2a$ and $b^2 = a^2 - c^2$. The eccentricity is $e = c/a$ and the ellipse has vertices at $(-a, 0)$, $(a, 0)$, $(0, b)$, and $(0, -b)$. This is calculated in Calculus 3 using the definition of an ellipse in terms of the sum of distances from the foci being constant (the second definition; Apollonius' Proposition III.52). Eccentricities and directrices are discussed in Calculus 3 in Section 11.7. Conic Sections in Polar Coordinates. Cartesian coordinates did not historically appear until René Descartes (March 31, 1596–February 11, 1650) introduced them in his *La Géométrie* which appeared as a supplement to his *Discours de la méthode* (1637). None-the-less, we take the equations as valid and extract some observations about ellipses based on the equations. An ellipse has two axes of symmetry, one is the line $\overleftrightarrow{FF'}$ and the other is the perpendicular bisector of segment FF'. The long axis of an ellipse is the *major axis* and it is of length $2a$ (assuming $a \geq b$), which is the constant sum of distances from the foci. The short axis is the *minor axis* and is of length $2b$.

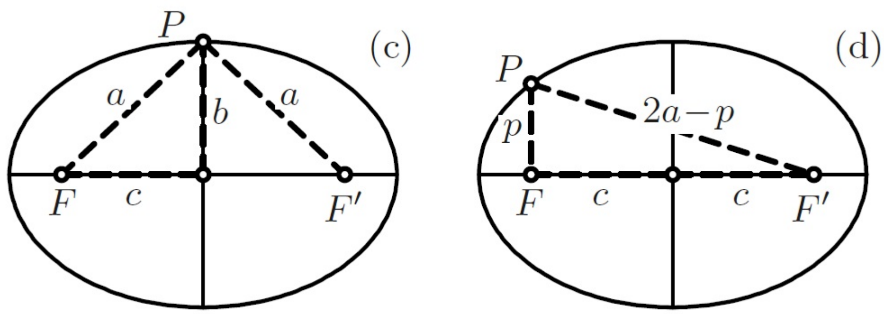


Fig. 3.5. An ellipse and its foci

With point P on the second symmetry axis as shown in Figure 3.5(c), we see that $b^2 = a^2 - c^2$ or $c^2 = a^2 - b^2$; we call b the *semi-minor axis* and a the *semi-major axis*. With point P vertically above F as shown in Figure 3.5(d), we have the *semi-latus rectum* p as the length of PF. Then $(2a - p)^2 = p^2 + (2c)^2$, which gives $a^2 - ap = c^2$ or $b^2 = ap$ (because $b^2 = a^2 - c^2$).

Note. As a passing observation, we note that Archimedes (287 BCE–212 BCE) proved that the area of the ellipse with semi-major axis of length a and semi-minor axis of length b is πab. He states this as Proposition 4 in his *On Conoids and Spheroids* as: "The area of any ellipse is to that of the auxiliary circle as the minor axis is to the major axis." The "auxiliary circle" is a circle of radius a containing the ellipse. So the auxiliary circle has area πa^2, and multiplying this by b/a ("as the minor axis to the major axis") gives and area of the ellipse of $(\pi a^2)(b/a) = \pi ab$.

Notice that the area of the ellipse is also $\pi/4$ times the area of the rectangle inscribed around the ellipse. Archimedes uses the method of exhaustion by which he shows that the area cannot be strictly greater than πab nor can it be strictly less than πab, so that it must be exactly πab. He does this by circumscribing polygons around and inscribing polygons in the auxiliary circle, then scaling them (in one direction) by a factor of b/a to produce circumscribed and inscribed polygons in ellipse. Since he can find the area of a polygon, then he can perform his calculations. See Figure 3.6.

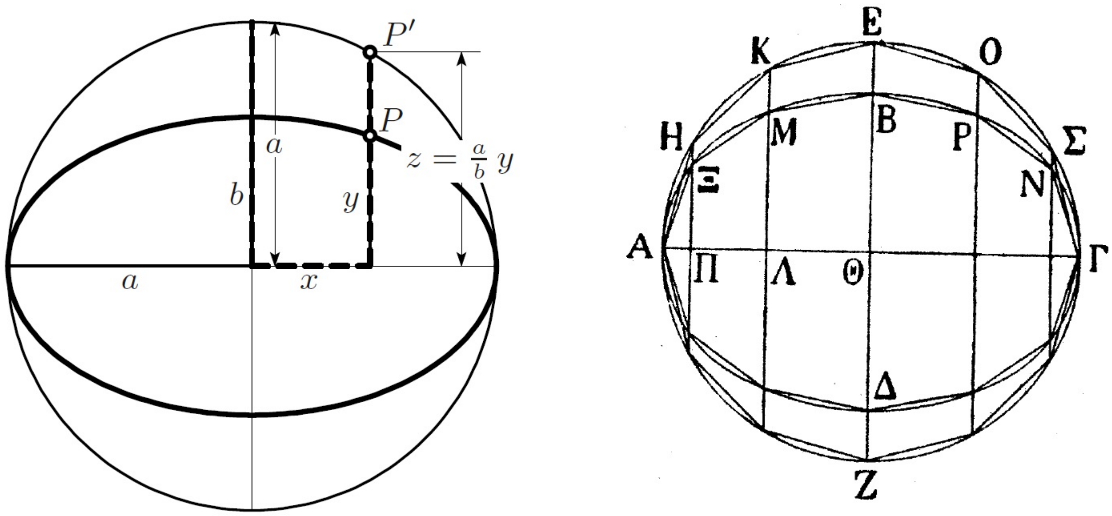


Fig. 3.6. Construction of an ellipse from a circle by a similarity transformation; right: drawing by Archimedes (*On conoids and spheroids*)

Archimedes takes a similar approach in his approximation of π as between $3\frac{10}{71}$ and $3\frac{1}{7}$; see my online presentation "Archimedes: 2,000 Year Ahead of His Time" in PowerPoint presentation along with a transcript in transcript of the presentation in PDF.

Note. Proclus gives a construction of an ellipse "with a stick." Consider segment (or "stick") DE of length $a + b$ which slides with its ends on the axes, as shown in

Figure 3.8 (right). The point P on the segment at a distance a from D and b from E will trace an arc of the ellipse with semi-axes a and b, respectively.

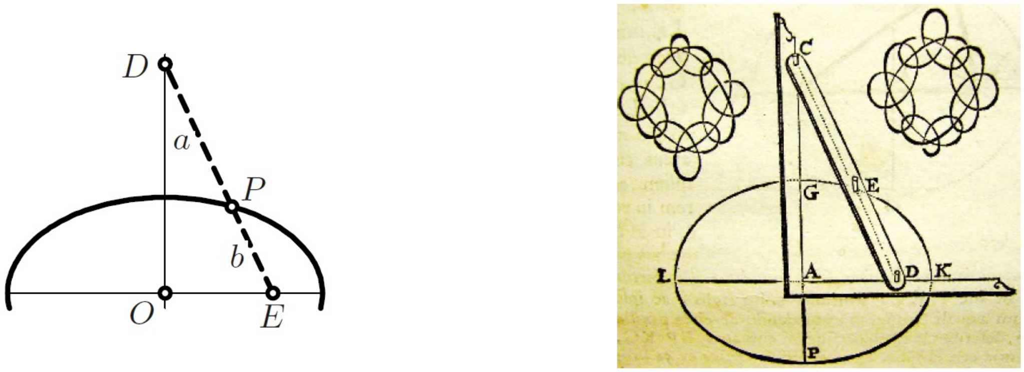


Fig. 3.8. (right) Construction of an ellipse with the help of a gliding stick together with illustration from van Schooten 1657

Note 3.2.D. Proclus' construction of an ellipse involves two concentric circles, one of radius a and one of radius b. The ray $\overrightarrow{OBA}$ rotates around point O producing points A and B. Point P is determined by projecting point A vertically and point B horizontally, as given in Figure 3.7(a).

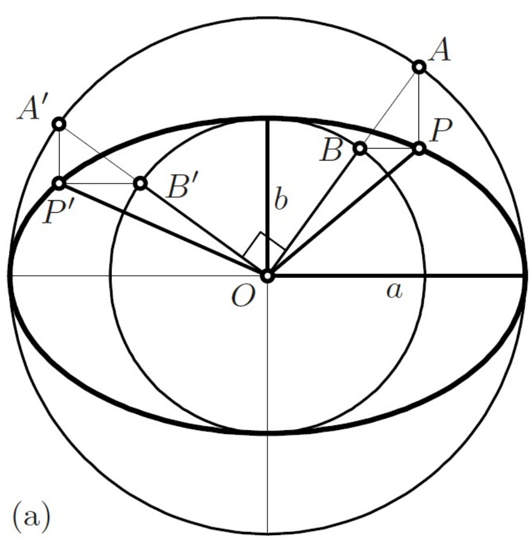


Fig. 3.7.(a) Construction of an ellipse by Proclus

We represent the circles parametrically in terms of angle θ as $(a\cos\theta, a\sin\theta)$ and $(b\cos\theta, b\sin\theta)$. Then the points P are of the form $(x, y) = (a\cos\theta, b\sin\theta)$. Therefore all points satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(a\cos\theta)^2}{a^2} + \frac{(b\sin\theta)^2}{b^2} = \cos^2\theta + \sin^2\theta = 1.$$

Therefore, the points P lie on the ellipse. In fact, $(a\cos\theta, b\sin\theta)$ is a parameterization of the ellipse. This construction can also be used to construct tangents to an ellipse. The two diameters of the circle of radius a resulting from line segments OA and OA' yield the diameters of the ellipse that result from the corresponding line segments OP and OP' in Figure 3.7(a). Such diameters of the ellipse are called *conjugate diameters* of the ellipse (they are not themselves orthogonal, unless they coincide with the diameters of the circle). Apollonius' Proposition II.6 in his *Treatise on Conic Sections* shows that each diameter is parallel to the tangents at the endpoints of its conjugate diameter and cuts its conjugate diameter in the point O. That is, a tangent to the ellipse of Figure 3.7(a) at point P' is parallel to OP (and a tangent at point P is parallel to OP'). See Figure 3.7(b).

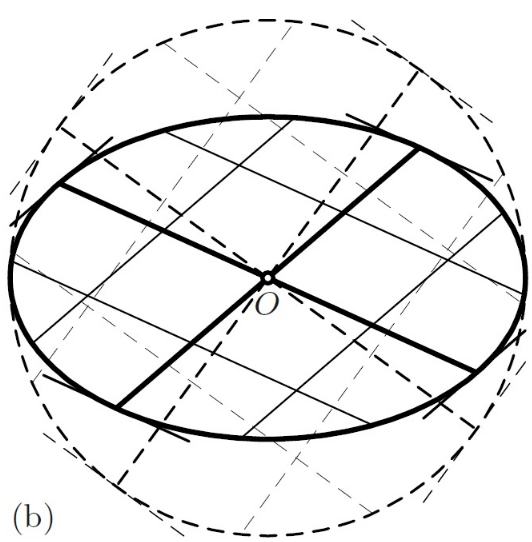


Fig. 3.7.(b) Construction of an ellipse by conjugate diameters

Note. In Note 3.2.D we saw how to find conjugate diameters of an ellipse (from which we can find tangents to the ellipse at a given point). We now consider the inverse problem of given two conjugate diameters of an unknown ellipse to find the semi-axes of the ellipse. The construction starts by rotating one semi-diameter through a right angle. Semi-diameter OP' (along with triangle $\triangle A'P'B'$; we do not know this triangle since we do not already know the semi-diameters a and b) of Figure 3.7(a) (see below) is rotated through a right angle in the clockwise direction. This gives the segment OQ (and triangle $\triangle AQB$; again, unknown) of Figure 3.8. In this way, the two unknown right triangles meet up on their hypotenuses to form a rectangle. (It seems that we now know that Q and P are opposite corners of a rectangle and that we can next find points A and B that determine the semi-diameters a and b, similar to the horizontal and vertical projections that allowed us to find point P in Note 3.2.D, but a different approach is given in Ostermann and Wanner.) Define point M as the midpoint between points P and Q. We now know that M is at a distance of $(a+b)/2$ from point O. Draw circle C with center M that passes through point O. We now know that this circle has radius $(a+b)/2$. Define point E as the other point of intersection of circle C and the horizontal line through O, and define point D as the point opposite E on C. We now know that $QP = a - b$ (it equals AB) and $MD = ME = (a+b)/2$, so that $PE = ME - MP = ME - \frac{1}{2}QP = (a+b)/2 - (a-b)/2 = b$. Since $PD = PM + MD = \frac{1}{2}PQ = (a+b)/2 + (a-b)/2 = a$. So this construction gives the semi-axes of the ellipse as $PE = b$ and $PD = a$.

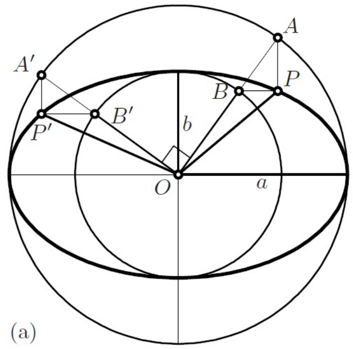


Fig. 3.7.(a) Construction of an ellipse by Proclus

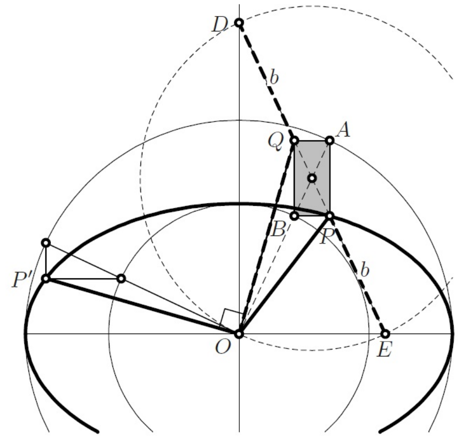


Fig. 3.8. Rytz' construction of an ellipse from two conjugate diameters

Parts of this construction are due to Euler and Frézier (the idea of rotating the semi-diameter), but Ostermann and Wanner say that it is attributed to "Daniel Rytz" since 1845. However, Wikipedia reports that the proof is due to "David Rytz von Brugg" (April 1, 1801–March 25, 1868). See the Wikipedia pages on David Rytz and Rytz's Construction (accessed 5/6/2023). A minor point...

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