Section 3.3. The Hyperbola.

Note. In Section 3.1. The Parabola we considered the intersection of a plane π with a cone where the plane is parallel to one of the generators of the cone (see Figure 3.2 in the supplement concerning the proof of Theorem 3.1. In Section 3.2. The Ellipse we considered the intersection of a plane π with a cone with a plane that is "less steep" than a generator of the cone (see Figure 3.3). In this section we consider the intersection of a plane π with a cone that is "steeper" than a generator of the cone. Unlike in Sections 3.1 and 3.2, in this section we must consider a "double cone" generated by a line (instead of a "single cone" generated by a ray as in Sections 3.1 and 3.2). See Figure 3.10.

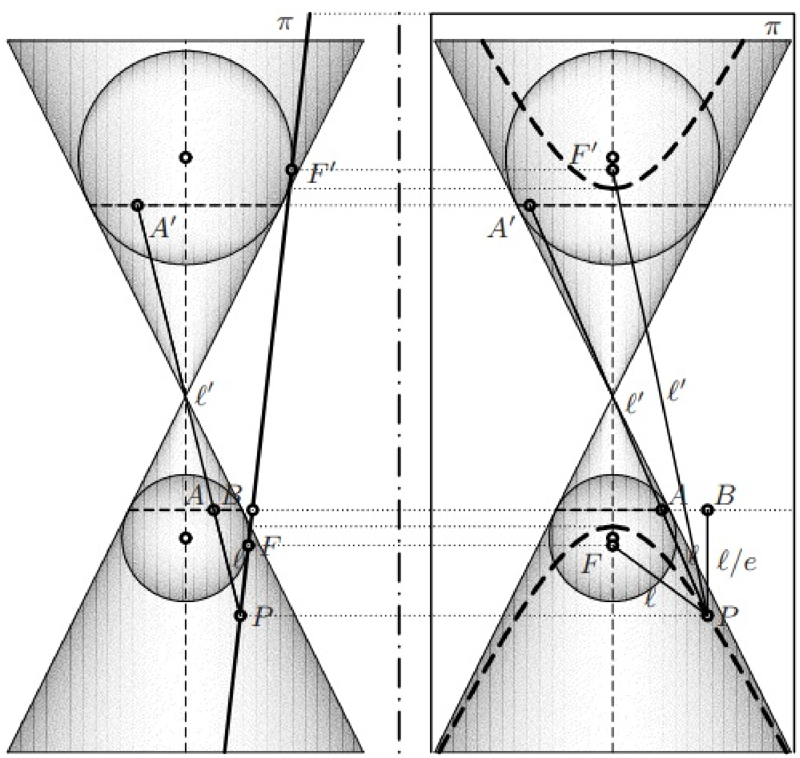


Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Definition. A *hyperbola* is the locus of all points in a plane whose distances from two fixed points in the plane (called foci) have a constant difference. (See Figure 3.11 where the constant difference is given by $\ell' = \ell$.)

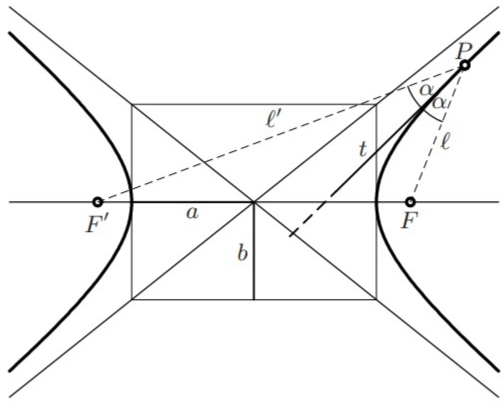


Fig. 3.11. A hyperbola and its tangent

Note. The next result shows that the intersection of a plane (of certain steepness) with a double cone produces a hyperbola. This allows us to connect Menaechmus' approach to hyperbolas with the idea of a locus of points whose distances from two fixed points (foci) have a constant difference. We discuss below how these ideas are combined with the directrix and focus approach given above. The result appears in Apollonius' *Treatise on Conics* as Proposition III.51. The proof is very similar to that of Theorem 3.2.A (Apollonius' Proposition III.52). The proof uses Dandelin spheres, as did Theorems 3.1 and 3.3.A. In fact, the next result appears as Exercise 8.10 of Keith Kendig's *Conics*, The Dolciani Mathematical Expositions #29, Mathematical Association of America (2005).

Theorem 3.3.A. (Apollonius' Proposition III.51 in *Treatise on Conic Sections*) The intersection of a double cone and a plane that is more steep than the generators of the cone is a locus of all points in a plane whose distances from two fixed points in the plane (called foci) have a constant difference.

Note. Notice that we can also define a hyperbola in terms of a directrix and a focus. Consider the line resulting from the intersection of the plane containing circle C and plane π as the directrix d and point F, as in Figure 3.10. We see that if the distance of any point P on the lower branch of the hyperbola is ℓ, then the distance from P to the directrix d is ℓ/e. So the the ratio of the distances of points on the hyperbola to the point F and to the line d equals $e > 1$. See Figure 3.10 again. This yields the following definition.

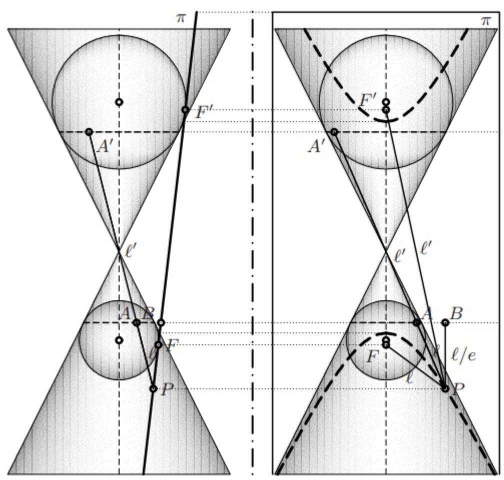


Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Definition. Let d be a line (called a *directrix*) and F a point (called a *focus*) at a distance p from d. A *branch of a hyperbola* is the locus of all points P (in the plane containing d and F) such that the ratio of the distances from point P to point F and from point P to line d equals constant $e > 1$ (called the *eccentricity*).

Note 3.3.A. We now have three definitions of a hyperbola:

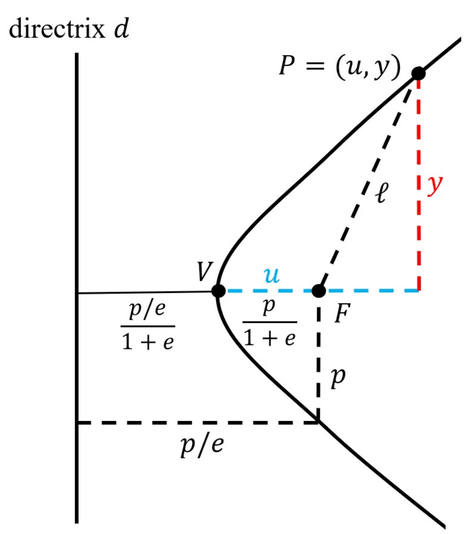
1. as the intersection of a cone and a "less steep" plane (Menaechmus' definition),

2. in terms of two fixed points, foci, and the difference of distances being constant (Apollonius' Proposition III.51), and

3. in terms of a directrix and focus (Pappus' definition).

Each of these are seen to be equivalent by Theorem 3.3.A. In Calculus 3 (MATH 2110; see my online notes for Calculus 3 on Section 11.6. Conic Sections), the definition if terms of two foci is used and it is from this that the standard form for the equation of a hyperbola is derived.

Note. Define the point on the branch of the hyperbola which is closest to the focus as the *vertex*, which we denote V. We proceed with the same computations as used for the ellipse in Section 3.2. The Ellipse. Notice that the point V must be a distance of $p/(1+e)$ from focus F, and V must be a distance of $(p/e)/(1+e)$ from the directrix d, so that the ratio of the distance from V to F to the distance from V to d is $\dfrac{p/(1+e)}{(p/e)/(1+e)} = e$, as required. See the figure below.



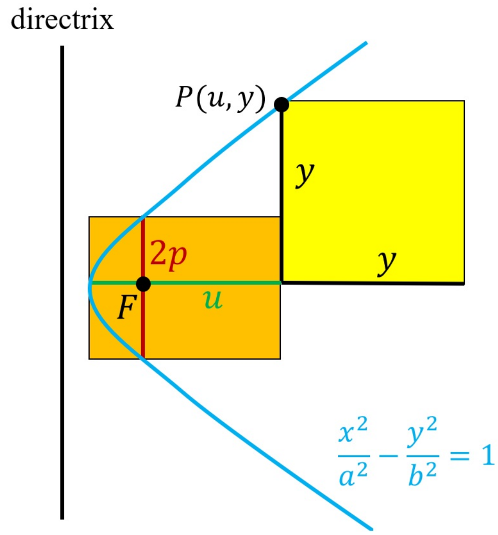
Let point $P = (u, y)$ be on the hyperbola in the coordinate system with its origin at the vertex and with the u-axis horizontal. We consider the right triangle with vertices P and F and hypotenuse given by the line segment PF. The legs of the triangle are then of lengths $u - p/(1+e)$ and y, and we let the hypotenuse have length ℓ. Notice that we then have that point P is a distance of ℓ/e from directrix d, and that $\dfrac{\ell}{e} = u + \dfrac{p/e}{1+e}$ and so $\ell = e\left(u + \frac{p/e}{1+e}\right)$. So by the Pythagorean Theorem we have

$$\left(u - \frac{p}{1+e}\right)^2 + y^2 = \ell^2 = e^2\left(u + \frac{p/e}{1+e}\right)^2 = \left(eu + \frac{p}{1+e}\right)^2,$$

$$\text{or } u^2 - \frac{2up}{1+e} + \frac{p^2}{1+e^2} - e^2u^2 - \frac{2eup}{1+e} - \frac{p^2}{1+e^2} + y^2 = 0,$$

$$\text{or } u^2 - \frac{2up(1+e)}{1+e} - e^2u^2 + y^2 = 0 \text{ or } y^2 = 2up + (e^2-1)u^2. \qquad (3.11)$$

Note 3.3.B. As the following figure shows, any point $P(u, y)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ determines a rectangle of width u and height $2p$, and a square of width y and height y. Since the rectangle has area $2up$ and the square has area y^2, then the area of the square is greater than the area of the rectangle because $y^2 = 2up + (e^2 - 1)u^2$ (where $e > 1$) by (3.11). As stated on page 70 of Ostermann and Wanner, according to Henry Liddell and Robert Scott's Greek-English Lexicon (Oxford Press, 2016), "hyperbola" (in Greek, ὑπερβολή) means overshooting or excess. The inequality given by the fact that the area of the square is more than the area of the rectangle motivated Apollonius of Perga (circa 262 BCE–circa 190 BCE) to name the curve a *hyperbola*.



Note. Paraphrasing Apollonius himself, from Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections*, Edited in Modern Notation (Cambridge University Press, 1896), we have the following (where we have changed his labels of points so that they agree with the above picture; see Apollonius' page 10):

> It follows that the square on the ordinate (that is, the square with area y^2) is greater than a rectangle whose height is equal to the latus rectum $2p$ and whose base is the abscissa x. [Apollonius actually speaks of the equality of the square and a rectangle whose base overlaps the square by a certain amount. In this way he is dealing with an equation of the form $y^2 = px + (p/d)x^2$.] Hence the section is called a Hyperbola.

Note. With the axes of a coordinate system oriented horizontally and vertically in Figure 3.11, we find that x and y coordinates of points on the hyperbola are related as:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{3.13}$$

where a and b are as labeled in Figure 3.11.

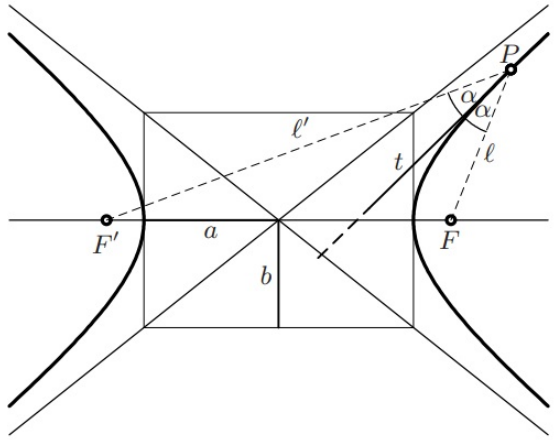
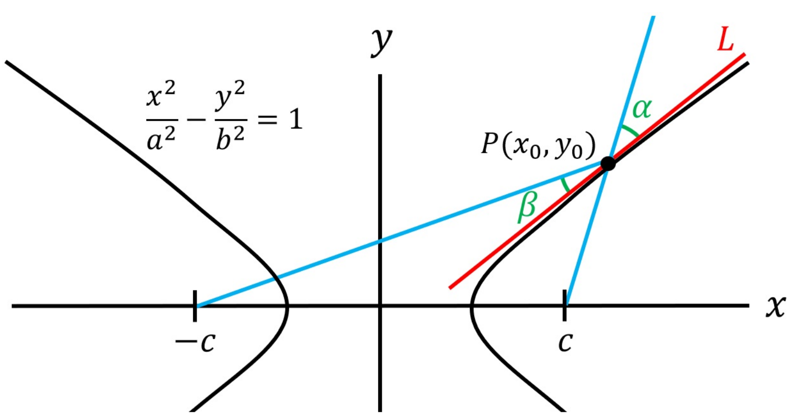


Fig. 3.11. A hyperbola and its tangent

The foci are a distance c from the center of the hyperbola where $c^2 = a^2 + b^2$ and the asymptotes are $\dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$ (asymptotes are discussed more below)

Note 3.3.C. The acute angle θ between two nonperpendicular, nonvertical intersecting lines of slopes m_1 and m_2 satisfies the equation $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$. This follows from the summation formula for tangent, where the absolute value is needed since the angle is required to be acute (the angle "between" two lines does not have an orientation, but if we interchange m_1 and m_2 then $\tan\theta$ changes by a negative sign).



Denote the foci as $F_1(-c,0)$ and $F_2(c,0)$ where $c>0$. Differentiating (implicitly) the formula for the hyperbola gives $\dfrac{2x}{a^2}-\dfrac{2y}{b^2}\dfrac{dy}{dx}=0$ or $\dfrac{dy}{dx}=\dfrac{2x/a^2}{2y/b^2}=\dfrac{xb^2}{ya^2}$. So the slope of line L in in the above figure is $m_L=\left.\dfrac{dy}{dx}\right|_{(x,y)=(x_0,y_0)}=\dfrac{x_0b^2}{y_0a^2}$. The slope of $\overleftrightarrow{F_1P}$ is $m_{\overleftrightarrow{F_1P}}=\dfrac{y_0-0}{x_0-(-c)}=\dfrac{y_0}{x+c}$. The slope of $\overleftrightarrow{F_2P}$ is $m_{\overleftrightarrow{F_2P}}=\dfrac{y_0-o}{x_0-c}=\dfrac{y_0}{x_0-c}$. We now have $\tan\alpha$ as

$$\begin{aligned}
\tan\alpha &= \left|\frac{m_L-m_{\overleftrightarrow{F_2P}}}{1+m_Lm_{\overleftrightarrow{F_2P}}}\right| = \left|\frac{\frac{x_0b^2}{y_0a^2}-\frac{y_0}{x_0-c}}{1+\frac{x_0b^2}{y_0a^2}\frac{y_0}{x_0-c}}\right| \\
&= \left|\frac{x_0b^2(x_0-c)-y_0(y_0a^2)}{y_0a^2(x_0-c)+x_0b^2y_0}\right| = \left|\frac{x_0b^2-x_0b^2c-y_0^2a^2}{x_0y_0a^2-y_0a^2c+x_0y_0b^2}\right| \\
&= \left|\frac{a^2b^2-x_0b^2c}{x_0y_0(a^2+b^2)-y_0a^2c}\right| \text{ since } \frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1 \text{ so that } x_0^2b^2-y_0^2a^2=a^2b^2 \\
&= \left|\frac{b^2(a^2-x_0c)}{y_0c(x_0c-a^2)}\right| = \left|\frac{-b^2}{y_0c}\right| \text{ since } c^2=a^2+b^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tan\beta &= \left|\frac{m_L-m_{\overleftrightarrow{F_1P}}}{1+m_Lm_{\overleftrightarrow{F_1P}}}\right| = \left|\frac{\frac{x_0b^2}{y_0a^2}-\frac{y_0}{x_0+c}}{1+\frac{x_0b^2}{y_0a^2}\frac{y_0}{x_0+c}}\right| \\
&= \left|\frac{x_0b^2(x_0+c)-y_0(y_0a^2)}{y_0a^2(x_0+c)+x_0b^2y_0}\right| = \left|\frac{x_0b^2+x_0b^2c-y_0^2a^2}{x_0y_0a^2+y_0a^2c+x_0y_0b^2}\right| \\
&= \left|\frac{a^2b^2+x_0b^2c}{x_0y_0(a^2+b^2)+y_0a^2c}\right| \text{ since } \frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1 \text{ so that } x_0^2b^2-y_0^2a^2=a^2b^2 \\
&= \left|\frac{b^2(a^2+x_0c)}{y_0c(x_0c+a^2)}\right| = \left|\frac{b^2}{y_0c}\right| \text{ since } c^2=a^2+b^2.
\end{aligned}$$

Therefore $\tan\alpha=\tan\beta$ and, since α and β are acute, then $\alpha=\beta$. If a ray of light is approaching focus F_2 and reflects off of the right branch of the hyperbola at point P, the the angle of incidence is α and since $\alpha=\beta$ then the angle of reflection is β (by the Law of Reflection) and hence the light will travel to the other focus at F_2. This is the reflective property of the hyperbola.

Note. We conclude the topic hyperbola with a discussion of asymptotes. Consider the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This can be factored to give $\left(\frac{x}{a} + \frac{y}{b}\right)\left(\frac{x}{a} - \frac{y}{b}\right) = 1$. With x and/or y "large," the left hand side of this equation becomes large so that, as a consequence, the right hand side becomes relatively small. So, in the limit, we get the terms of the left hand side approaching zero (this is Ostermann and Wanner's argument of page 71), This implies that the asymptotes at then $y = (b/a)x$ and $y = -(b/a)x$. Also on page 71 of Ostermann and Wanner, Henry Liddell and Robert Scott's *Greek-English Lexicon* (Oxford Press, 2016) is referenced for defining "symptosis" (in Greek, $\sigma\acute{\nu}\mu\pi\tau\omega\sigma\iota\varsigma$) as "falling together, collapsing, meeting." So one could take "asymptote" to mean that there is no meeting. This is advocated by Apollonius himself in Proposition II.1 of his *Treatise on Conic Sections* where he finds the asymptotes of a hyperbola and states that if the asymptotes are "produced will not meet the curve in any finite point and are accordingly defined as asymptotes" (see page 53 of Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections*, Edited in Modern Notation, Cambridge University Press, 1896). This is not the modern, more general sense in which the term "asymptote" is used. For example, the damped sine wave $y = e^{-x} \sin x$ satisfies $\lim_{x\to+\infty} e^{-x} \sin x = 0$ so that $y = 0$ is a (right) horizontal asymptote of the function, but the graph of the function *does meet* the asymptote; $e^{-x} \sin x = 0$ for all $x = n\pi$ where $n \in \mathbb{Z}$. This illustrates the, as conceived today, an asymptote is not something that the curve "gets closer and closer to, but does not get there." For a very brief discussion of this, see my small publication "Horizontal Asymptotes: What They are Not," *The Mathematics Teacher* (Reader Reflections), February 1998, 152, available online in PDF.

Note. We have seen that the key players in conic sections are Menaechmus (circa 380 BCE–circa 320 BCE), Apollonius of Perga (circa 262 BCE–circa 190 BCE), and Pappus of Alexandria (circa 290–circa 350). They have given us the parabola, ellipse, and hyperbola as (1) intersections of cones with planes (Menaechmus), (2) as determined by the relationship between the areas of a square a rectangle, and for the ellipse and hyperbola in terms distances from the two foci (Apollonius, who gives the curves their names as a result of the first of these relationships), and (3) in terms of a directrix, focus, and eccentricity (Pappus).

The frontispiece of Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections* (1896)