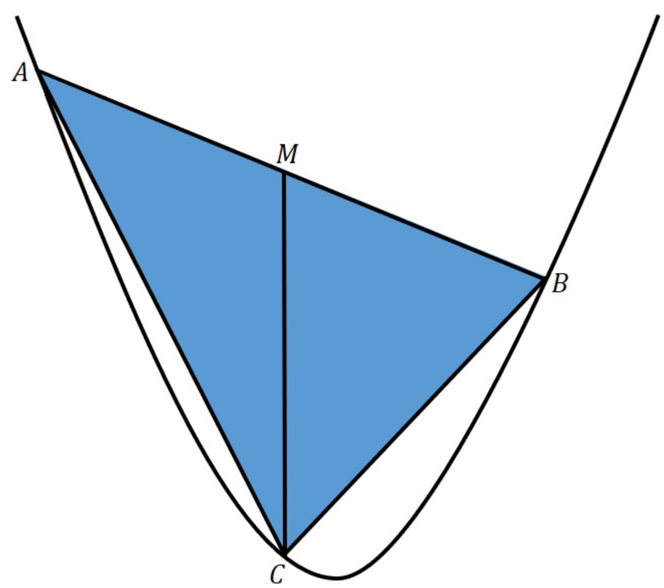
Section 3.4. The Area of a Parabola.

Note. Archimedes of Syracuse (287 BCE–212 BCE) wrote a letter to a friend (Dositheus), which later became known as his work *Quadrature of the Parabola*. In it, he gave two arguments related to the area bounded by part of a parabola. His result is that the area bounded by the parabola and line intersecting it in two points (determining a chord) is 4/3 of the area of a triangle with the chord as its base and the third vertex as the point on the parabola in the direction parallel to the axis from the midpoint of the chord. That is, in the following figure, the area bounded by segment AB and the parabola equals 4/3 of the area bounded by the triangle (this area is in blue), where M is the midpoint of segment AB.



In one argument (which we will explore in this section in more detail), he partitions the region into little triangles (the areas of which are easy to find) and sums these areas. This approach is purely geometric, but does require some concept of an infinite series. For a description of this approach (and related results), see Gordon

Swain and Thomas Dence's "Archimedes' Quadrature of the Parabola Revisited," *Mathematics Magazine*, **71**(2) (April 1998), 123–130 (1998). In Archimedes second argument (as described by Swain and Dence on their page 124) "...the segment is divided into wedges with a common vertex at one end of the chord, and the formula is found through center of mass arguments and increasing the number of wedges." Swain and Dence also comment in a footnote that the Arab mathematician Thābit ibn Quarra (836–February 18, 901) gave a proof based on dividing the segment into slices parallel to the chord, "giving a Riemann integral style derivation of the area." This gives Thābit ibn Quarra a place in the history of integration; Riemann's work on integration dates from the mid 1800s, so Thābit's work predates Riemann's by almost 1000 years.

Thābit ibn Quarra from MacTutor History of Mathematics Archive webpage on Al-Sabi Thābit ibn Qurra al-Harrani

Note. Archimedes gives another argument about the area bounded by a parabola in *The Method*. In Proposition 1 of *The Method* he uses an idea of partitioning the area into individual line segments, balancing the segments using a fulcrum (where a segment is "weighted" by its length), and producing a rectangular area that balances with the area bounded by the parabola. Since the area of a rectangle is easy to calculate, then the area bounded by the parabola can be calculated. Details for this argument are given in Reviel Netz and William Noel's *Archimedes Codex, How a Medieval Prayer Book is Revealing the True Genius of Antiquity's Greatest Scientist* (Da Capo Press, 2007) (see pages 150–157). Netz and Noel also comment "*The Method*, by bringing together mathematics, physics, and infinity, raised the most fundamental question of science. It anticipated Newtons calculus..." (page 157). More on this can be found in my my online PowerPoint presentation (based in great part on Netz and Noel's book) on Archimedes: 2000 Years Ahead of His Time and the transcript in PDF (notice the section on "Archimedes and Integration"). This presentation also includes details on Archimedes computations which show that $3\frac{10}{71} < \pi < 3\frac{1}{7}$ (and the classical approximation of π as $\approx 22/7$).

Note. We now return to Archimedes argument based on summing areas of triangles. For simplicity, we consider a simplified case where the chord is perpendicular to the axis of the parabola, as given in Figure 3.12. Notice that in this case, the triangle to which Archimedes refers is an isosceles triangle. Let the area of this triangle be $\mathcal{T}$, where the base is $2b$ and the height is a. With $\mathcal{P}$ as the area under the parabola, Archimedes claim is then that $\mathcal{P} = \frac{4}{3}\mathcal{T}$.

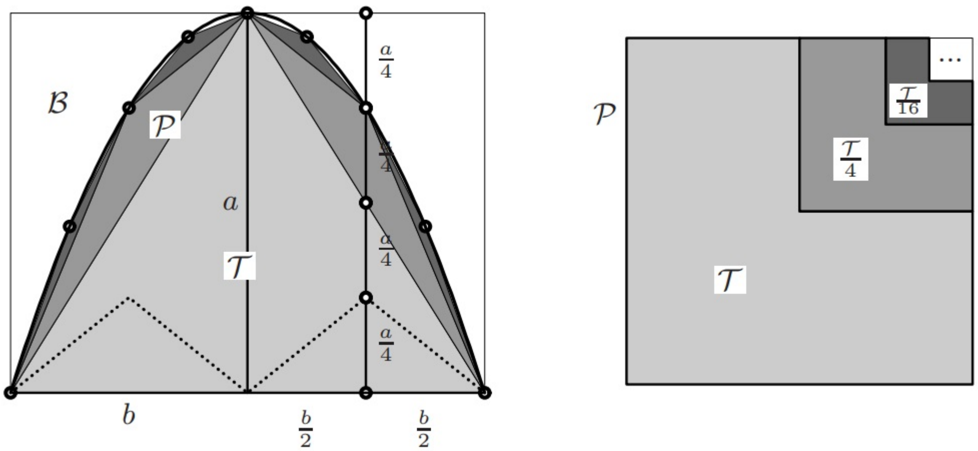
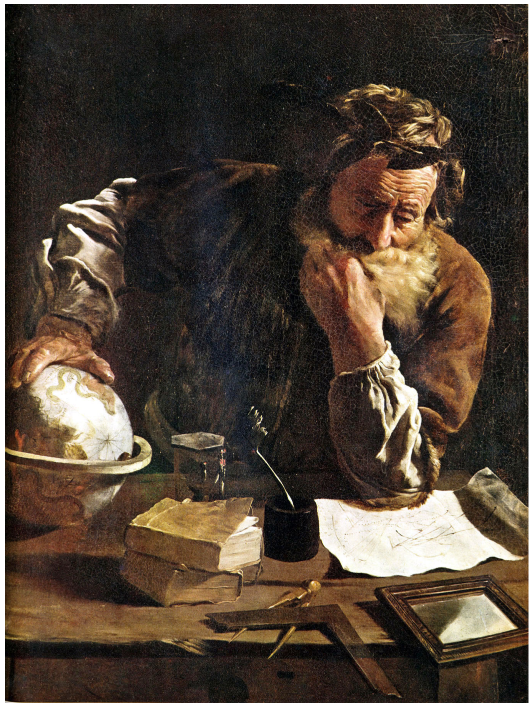


Fig. 3.12. The quadrature of the parabola

Theorem 3.4.A. With $\mathcal{P}$ as the area under the parabola given in Figure 3.12 (left) and with $\mathcal{T}$ as the area of the large isosceles triangle, we have $\mathcal{P} = \dfrac{4}{3}\mathcal{T}$.

Note. Plutarch (circa 46 CE–circa 119 CE) provides an account of Archimedes' death at the hands of a Roman soldier during the 212 BCE Roman capture of Syracuse in the Second Punic War. Archimedes is said to have been working on math when the soldier entered his quarters with sword drawn. Archimedes asked to be left alone so that he could complete his work and the soldier stabbed him to death. A Greek author, Johannes Tzetzes, writes (in the twelfth century) that Archimedes dies at the age of 75, and that's where the birth year of 287 BCE arises.

Over the centuries, a number stories have been told about Archimedes. One is that he stepped into a full bathtub and noticed the water overflowing as he entered the tub. He realized that he could measure the volume of the water and use this to determine the volume of his body. This story told by Vitruvius about two centuries after Archimedes' death and is an urban legend. The story seems to be a way to explain Archimedes use of displacement of water to measure the volume of a crown which was expected to contain lead instead of gold. The crown was easy to weight, but a measurement of its volume was a challenge due to the exotic shape.



Archimedes Thoughtful by Domenico Fetti (1620), from Wikipedia

A number of Archimedes' works have survived, including *On Plane Equilibriums*, *Quadrature of the Parabola*, *The Method*, *On the Sphere and Cylinder*, *On Floating Bodies*, and *Measurement of a Circle*. In *On Plane Equilibriums*, the center of mass of and object is considered. *Quadrature of the Parabola* addresses the problem mentioned in this section. *On Floating Bodies* addresses displacement of water and the relation of this volume to the mass of the body. In *On the Sphere and Cylinder*, Archimedes uses a cylinder circumscribed around a sphere to show that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

In *Measurement of a Circle*, Archimedes proves the circumference of a circle is $C = 2\pi r$. First, Euclid's Book XII, Proposition 2 states that the area of a circle if proportional square of its radius. The constant of proportionality is π, so that $A = \pi r^2$. In Proposition 1 of *Measurement of a Circle*, Archimedes proved that the constant of proportionality π relating the radius to the area of a circle also relates the circumference and radius of a circle. The proof has two parts. In the first part, Archimedes assumes that a relevant quantity is greater than what is desired and he gets a contradiction. In the second part, Archimedes assumes that a relevant quantity is less than what is desired and he gets another contradiction. So that equality must be the case. (The "relevant quantity" is actually the area of a circle; the circumference enters the conversation as the leg of a right triangle. As with much of ancient Greek geometry, quantities such as areas are not given numerical values as much as they are described in terms of areas of known shapes.) In his argument, he inscribes regular polygons in a circle and uses these to compute areas. Also in *Measurement of a Circle*, Archimedes uses inscribed and circumscribed regular polygons and their perimeters to show $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

The Method is actually a letter from Archimedes to Eratosthenes. In his other works, Archimedes often deals with the difference between two quantities (one known, and one suspected) always becoming smaller than any given magnitude. This is what you encounter today as an ε argument in calculus or analysis (which are ideas first rigorously explained by Augustin Cauchy in the 1800s). This is what philosophers refer to as "potential infinity." *The Method* has an interesting story in how it reached us today. The copy that survives was originally written on sheep skin. Later, the sheep skin was scrapped clean, cut, rotated 90°, and written over to create a medieval prayer book; this was a common process and the resulting book is called a *palimpsest*. In reconstructing *The Method* it was realized in 2001 that

Archimedes knew of actual infinity and used it in his mathematics. So Archimedes was 1000s of years ahead of his time; these ideas arise around the year 1700 with the work of Newton and Leibniz. As mentioned, it was not rigorously cleaned up until Cauchy's work in the 1800s.

This biographical information is from Reviel Netz and William Noel's *Archimedes Codex, How a Medieval Prayer Book is Revealing the True Genius of Antiquity's Greatest Scientist* (Da Capo Press, 2007). The Public Broadcasting System presented an episode of NOVA titled *Infinite Secrets: The Genius of Archimedes* in 2004. There is also a PBS resource page on *Infinite Secrets*, which includes a transcript of the program.

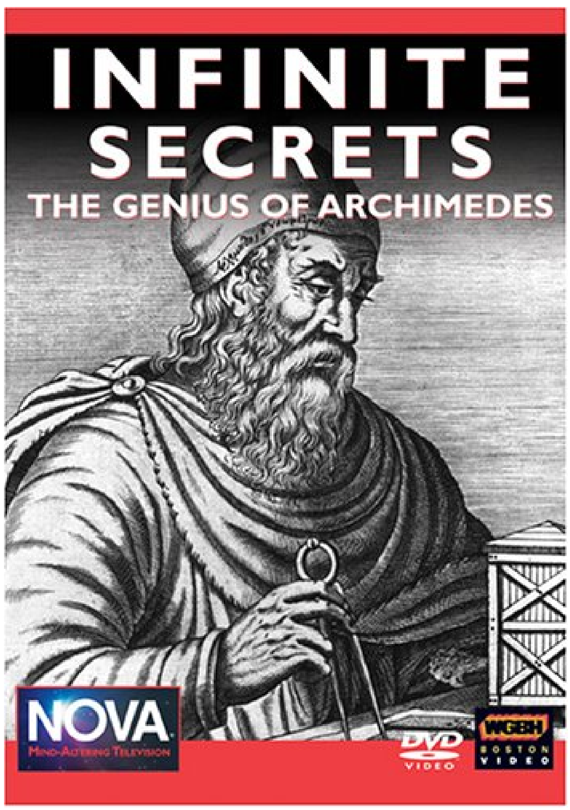


Image from Amazon.com (accessed 12/8/2021)