Chapter 4. Further Results in Euclidean Geometry

Section 4.1. The Conchoid of Nicomedes, The Trisection of an Angle.

Note. Very little is known about the life of Nicomedes. What is known is largely based on Pappus of Alexandria's (circa 290–circa 350) *Synagoge* or *Mathematical Collection*, and eight book collection of mathematical writings from around 340 CE. In Book IV he addresses trisection techniques, the quadratrix of Hippias, and the spiral of Archimedes (which we consider in the next section). Pappus comments that Nicomedes introduced the conchoid and used it to trisect and angle. Pappus also credits Nicomedes with using the quadratrix of Hippias to square the circle. This history is based on the MacTutor History of Mathematics Archive biographies on Nicomedes and Pappus (see also Thomas Heath's *A History of Greek Mathematics*, Oxford: Clarendon Press. 1921).

Nicomedes (circa 280 BCE–210 BCE)

Image from MacTutor History of Mathematics Archive biography of Nicomedes

(accessed 11/22/2021)

Note. While exploring the doubling of the cube (one of the three unsolved compass and straightedge constructions problems of ancient Greece; see Section 1.8. Three Famous Problems of Greek Geometry), Nicomedes defined the conchoid. In Exercise 6.10.2, you are asked to show that this construction can in fact be done. We present the conchoid here, however, to solve another of the three famous problems: the trisection of the angle.

Definition. Let A be a fixed point, let ℓ a line at a distance c from A, and let b be a given positive value. For any line ℓ_D through point A that is not parallel to ℓ, let D be the point of intersection of lines ℓ_D and ℓ. Define point C to be a point on line ℓ_D that a distance b from point D and on the opposite side of line ℓ as point A. The collection of all such points C and ℓ_D ranges over all line through point A that are not parallel to ℓ is a *conchoid*. We can also take point C to be on the same side of ℓ as point A to produce a conchoid.

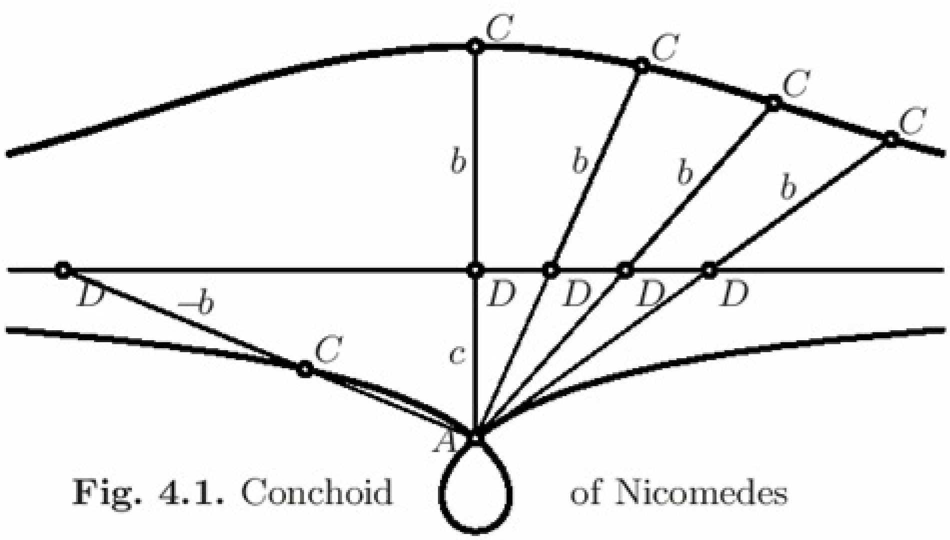


Fig. 4.1. Conchoid of Nicomedes

Note. Notice that for points C on the opposite side of line ℓ from point A, the conchoid is smooth, with no loops (see Figure 4.1, top). But for points C on the same side of line ℓ as point A, the conchoid may have a cusp or a loop (see Figure 4.1, bottom). If line ℓ is of the form $x = a$ (so, unlike Figure 4.1, the line is vertical) then its has the polar coordinate representation $r = a \sec \theta$ and the parametric representation of the conchoid is $x = \pm d \cos \theta$, $y = a \tan \theta \pm d \sin \theta$. In Cartesian coordinates, the conchoid this concoid is $(x - a)^2(x^2 + y^2) = b^2x^2$. In both of these representations, we get both "lobes" of the conchoid; that is, we get points both on the same and opposite sides of ℓ as point A.

Note. It was Pappus of Alexandria who realized that the conchoid of Nicomedes can be use to trisect an angle. We start with angle α with vertex at point A, and point B on one ray of the angle where AB has length a (see Figure 4.2, left). Next construct a perpendicular to the other ray of the angle and through point B (Euclid, Book I Proposition 12), and let E be the point of intersection of the perpendicular with the ray (see Figure 4.2, right). Next, construct a parallel to AE through point B (Euclid, Book I Proposition 31). Consider the conchoid with point A as the fixed point, line BE as the line ℓ, and $2a$ as the distance b. Let point C be the intersection of the conchoid with line AD. The C is a distance of $2a$ from point D of intersection of line AC with line $\ell = BE$ (and on the opposite side of line $\ell = BE$ from point A). See Figure 4.2 right. We claim that angle EAD is $\alpha/3$, so that we have trisected arbitrary angle α. We prove this in the next theorem. Notice that Figure 4.2 requires that α is an acute angle. If α is any angle, then we can add or subtract multiples of $90°$ until we produce an acute angle (and $90°/3 = 30°$ can be constructed since a $60°$ angle can be construction because an equilateral triangle can be constructed by Euclid, Book I Proposition 1, and an angle can be bisected by Euclid, Book I Proposition 9), so we may assume without loss of generality that α is acute.

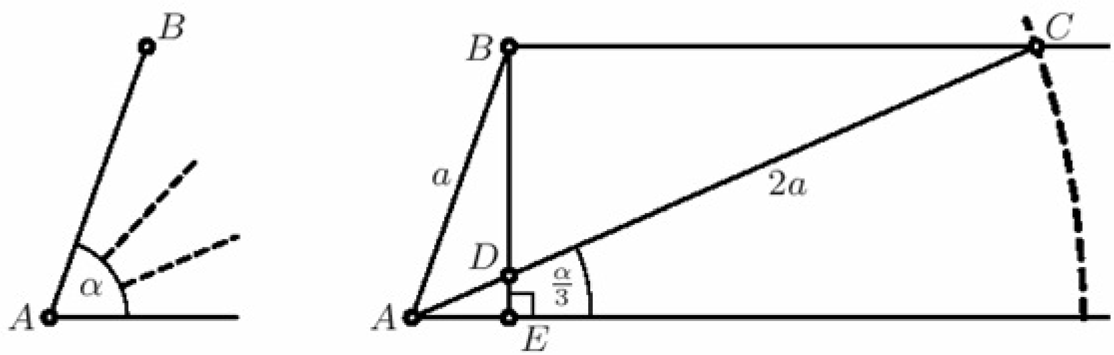


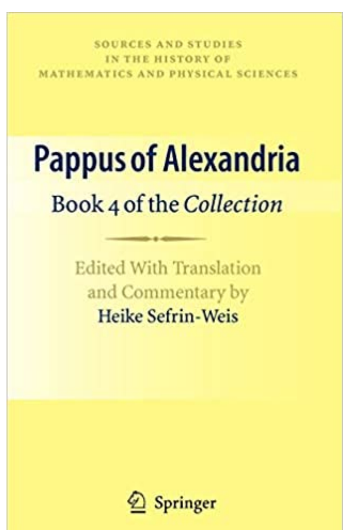
Fig. 4.2. Pappus' trisection of an angle with the help of the conchoid

Theorem 4.1.A. (Pappus' Proposition IV.32 in *Collection*)
Assuming the existence of the conchoid of Nicomedes, we can trisect any angle.

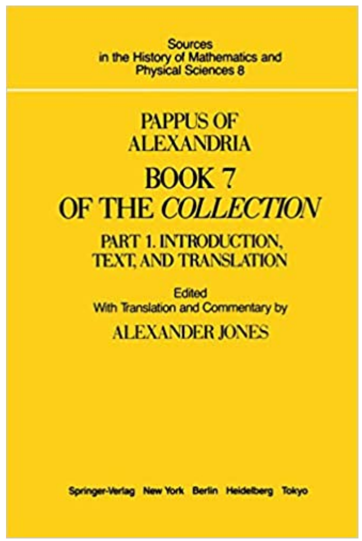
Note. Pappus also proved that the conic section of a hyperbola can be used to trisect an angle. He presents this also in Book IV of his *Synagoge* or *Mathematical Collection*. We should comment that these solutions to the "famous problems" go beyond the use of a compass and straightedge and so are not in the spirit of the setting in which the problems were original posed (and the setting in which impossibility is resolved using the field of constructible numbers, as discussed in Section 1.8. Three Famous Problems of Greek Geometry).

Theorem 4.1.B. (Pappus' Proposition IV.31 in *Collection*)
Assuming the existence of the hyperbola, we can trisect any angle.

Note. Pappus of Alexandria (circa 290–circa 350) is "the last of the great Greek geometers" according to the MacTutor History of Mathematics Archive biography of Pappus (from which the following history is based). As mentioned above, his *Mathematical Collection*, written around 340 CE, consists of eight books. Book I is now lost, but it covered arithmetic. Book II is partly lost but the remaining part deals with Apollonius's method for dealing with large numbers. Book III deals with constructing means, geometrical paradoxes, and inscribing polyhedra in spheres. Book IV is the most relevant to the conversation at hand and is described above. In fact, Book IV is still in print as *Pappus of Alexandria: Book 4 of the Collection*, Edited With Translation and Commentary by Heike Sefrin-Weis, Sources and Studies in the History of Mathematics and Physical Sciences (Springer, 2010).

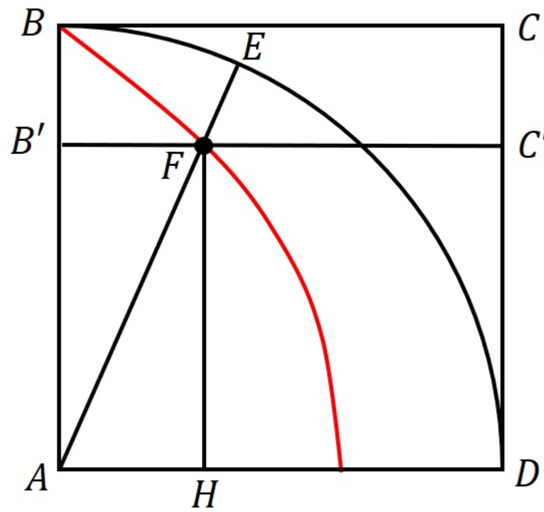


Book V describes hexagonal bee honeycombs and discusses the thirteen semiregular solds of Archimedes. Book VI deals with books on astronomy by others. Book VII considers the "Treasury of Analysis" and focuses on the work of Euclid, Apollonius, and Aristaeus the elder. Part 1 of this is in print as *Pappus of Alexandria Book 7 of the Collection: Part 1. Introduction, Text, and Translation* Edited and translated by Alexander Jones, Sources in the History of Mathematics and Physical Sciences 8 (softcover reprint of the original 1st edition, Springer-Verlag, 1986 Edition).

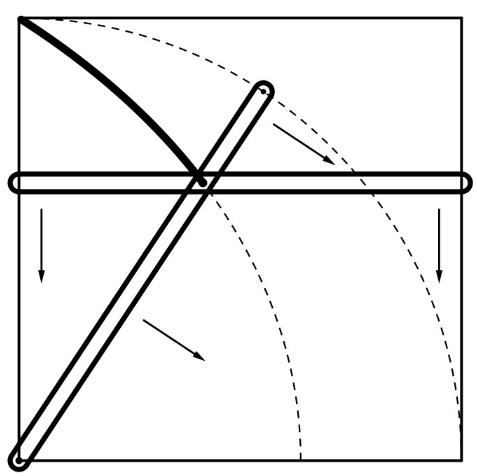


Book VIII deals with mechanics (movement of bodies about their centers of gravity, motion in space, and forces/collisions).

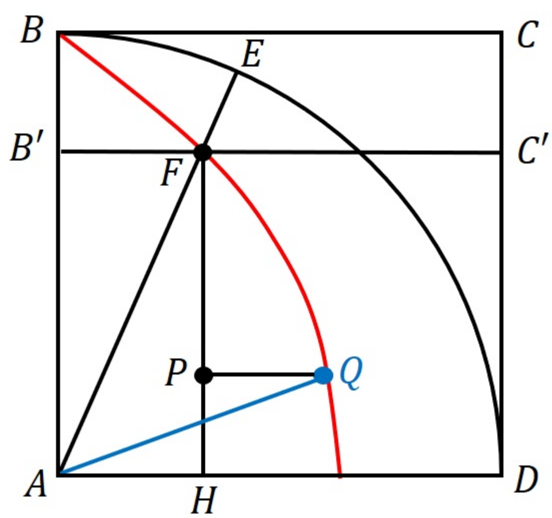
Note. Another key figure in the history of the trisection of an angle is Hippias of Elis (circa 460 BCE–circa 400 BCE). Pappus, in his Book IV, describes the quadtratrix of Hippias. The following description if from the MacTutor History of Mathematics Archive biography of Hippias. $ABCD$ is a square and BED is part of a circle, center A radius AB. As the radius AB rotates about A to move to the position AD then the line BC moves at the same rate parallel to itself to end at AD. Then the locus of the point of intersection F of the rotating radius AB and the moving line BC is the quadratrix.



With a as the length of the side of $ABCD$, we have the quadratrix as $f(x) = x \cot\left(\frac{\pi}{2a}x\right)$. The following image from the Wikipedia page for the Quadratrix of Hippias illustrates how the curve can be generated mechanically (the horizontal bar moves at a constant rate and the point of intersection of the two bars are joined by a pin that is free to slide along both bars).



Measuring angles in radians and with the uniform motion of $B'C'$, we have that $\frac{\text{angle } EAD}{\text{angle } BAD} = \frac{FH}{AB}$, or angle $EAD = \frac{FH}{AB}\frac{\pi}{2} = \frac{FH}{a}\frac{\pi}{2}$. With $AB = a = 1$, we simply have that angle EAD is proportional to the length of segment FH. So to trisect a given angle, say angle EAD in the figure above, we simply trisect segment FH and use the resulting point P on FH to find a corresponding point Q on the quadratix (using a horizontal line), which then determines the terminal side AQ of the angle of measure 1/3 that of angle EAD. Of course we can similarly find any rational multiple of a given angle (as explained in Section 1.2. Similar Figures; see Figure 1.6).



Pappus in his Book IV gives criticisms of the quadratrix (which he attributes to Sporus) concerning "the very thing for which the construction is thought to serve is actually assumed in the hypothesis." Sporus also criticizes the location of the point of intersection of the quadratrix with side AD of the square where, mechanically, the two bars coincide determining the quadratrix coincide (see Heath's *A History of Greek Mathematics, Volume 1: From Thales to Euclid*, Oxford: Clarendon Press [1921] pages 229–230).

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