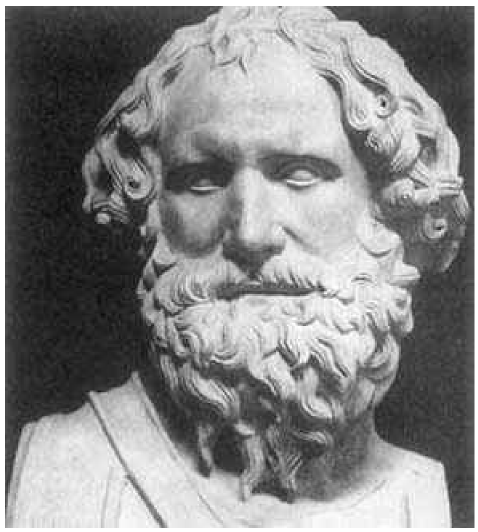
# Section 4.2. The Archimedean Spiral.

**Note.** According to E.T. Bell's *Men of Mathematics* (Simon and Schuster, 1937) page 20, "Any list of the three 'greatest' mathematicians of all history would include the name of Archimedes. The other two usually associated with him are Newton (1642–1727) and Gauss (1777–1855)." Archimedes of Syracuse (287 BCE–212 BCE) made substantial contributions to geometry (notice that he lived most of his life after the estimated death of Euclid [circa 325 BCE–circa 265 BCE]).



MacTutor History of Mathematics Archive biography of Archimedes (accessed 5/9/2023)

Using the method of exhaustion he established many of the formula for areas and volumes that you learn in high school (such as the area of an ellipse and the volume of a cone). In this way, he was very close to establishing integral calculus (and the Riemann integral) about 2,000 years before it formally entered the realm of mathematics! He was also reported to have designed various practical machines, such as the Archimedean screw for pumping water, and machines of war. We leave a detailed biography the History of Mathematics class (MATH 3040; see my online notes for that class on Section 6.2. Archimedes) and the supplement to this section, Archimedes: 2,000 Year Ahead of His Time (in PowerPoint, with a transcript available in PDF).

**Note.** In this brief section, we consider some of Archimedes' studies of tangents and areas of the Archimedean spiral (as it is called today). This worked is contained in his *On Spirals*, which has survived and is part of *The Works of Archimedes*, edited by Thomas Heath (Cambridge University Press, 1897). This is still in print by Dover publications; *On Spirals* appears on pages 151 to 188. On page 165, the following definition of Archimedes is given:

> "If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a *spiral* in the plane."

Ostermann and Wanner describe this in modern terms as (see page 81): "Consider a ray that rotates at constant angular velocity around the origin. Let $P$ be a point on the ray, that moves away from the origin at constant speed. Then the locus of $P$ is a curve called an *Archimedean spiral*, see Fig. 4.4, left [below]." Neither of these are rigorous mathematical definitions by today's standards, due to the reference to the informal ideas of movement and time. Of course Archimedes did not have access to Cartesian coordinates (due to René Descartes [March 31, 1596–February 11, 1650], who introduced them in his *La Géométrie* which appeared as a supplement to his *Discours de la méthode* in 1637) nor to polar coordinates. With polar coordinates

$(r, \theta)$ we can simply express the Archimedean spiral as the function $r = a\theta$ where $a$ is some constant (it represents the angular velocity of point $P$ in Ostermann and Wanner's description). Notice that Archimedes definition of spiral limits the polar coordinate definition to $\theta \in [0, 2\pi]$, since he only considers one revolution ("return to the position from which it started"), though we'll see below that he addresses "the $n$th turn" in his Proposition XX.

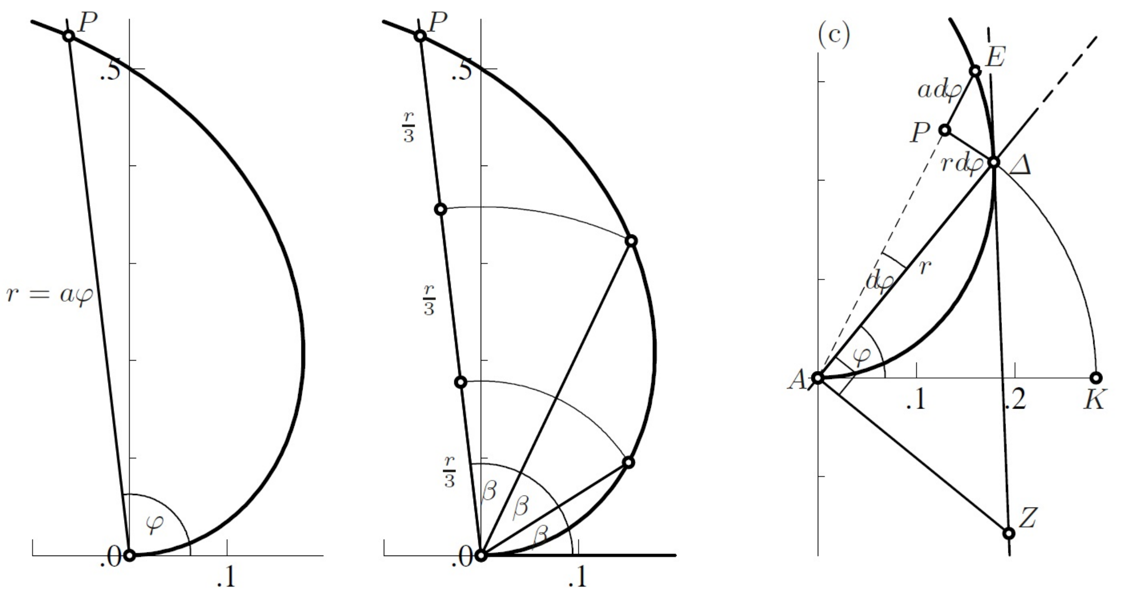
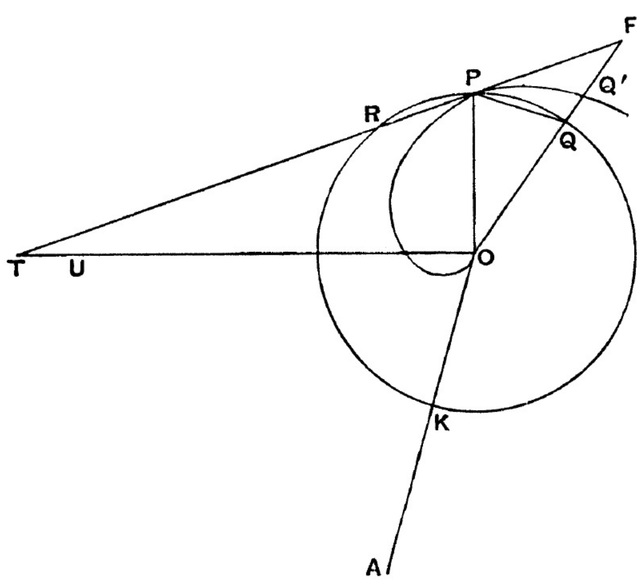



**Fig. 4.4.** Archimedean spiral for the trisection of an angle (middle), and its tangent (right)

**Note.** Notice that the Archimedean spiral allows us to trisect an angle, as given in Figure 4.4 (center). Simply trisect the terminal side (of length $r$ in the figure) of the given angle (of size $3\beta$) and use the resulting point to determine a circle of radius $r/3$. This circle intersects the Archimedean spiral at a point that determines an angle of size $\beta$, thus trisecting the given angle. Archimedes addresses a property of tangents to a spiral in his Proposition 20. Ostermann and Wanner state Proposition 20 as (see page 81): "That tangent at a point $\Delta$ cuts the line through $A$ orthogonal to $A\Delta$ at the point $Z$ such that $AZ = \text{arc } \Delta K$ (see Fig. 4.4(c))."

**Note.** Archimedes statement of Proposition 20 as given in Heath's *The Works of Archimedes* is:

> "**Proposition XX.** I. If $P$ be any point on the first turn of the spiral and $OT$ be drawn perpendicular to $OP$, $OT$ will meet the tangent at $P$ to the spiral in some point $T$; and if, the circle drawn with centre $O$ and radius $OP$ meet the initial line in $K$, then $OT$ is equal to the arc of this circle between $K$ and $P$ measured in the 'forward' direction of the spiral.
>
> II. Generally, if $P$ be a point on the $n$th turn, and the notation be as before, while $p$ represents the circumference of the circle with radius $OP$,
>
> $$OT = (n-1)p + \text{arc } KP \text{ (measured 'forward')."}$$




A figure from Heath's *The Works of Archimedes*, page 175. Notice that in this figure, the rotation is taken as clockwise, instead of the traditional counterclockwise of polar coordinates.

**Note.** With the differential notation of Leibniz, Archimedes takes a little increase of angle $\varphi$ of size $d\varphi$, so that the angle increases form $\varphi$ to $\varphi + d\varphi$. As shown in Figure 4.4(c), this gives a little right-angled triangle $\triangle \Delta PE$ with sides of approximate size $P\Delta \approx r\,d\varphi$ and $PE \approx dr = a\,d\varphi$ (because the curve is $r = a\varphi$ at point $\Delta$). For $d\varphi$ very small, we have angle $PE\Delta$ is approximately the same size as angle $A\Delta Z$. So for $d\varphi$ very small, triangle $\triangle \Delta PE$ is approximately similar to triangle $\triangle ZA\Delta$. Hence $(AZ)/(A\Delta) \approx (r\,d\varphi)/(a\,d\varphi) = r/a = \varphi$ (since $r = a\varphi$). Since $(A\Delta) = r$, then we now have $(AZ)/r \approx \varphi$, or $(AZ) \approx r\varphi =$ arc $\Delta K$. Now the smaller $d\varphi$ is, the better the approximation $(AZ) \approx$ arc $\Delta K$. "Therefore," in modern symbols, $\lim_{d\varphi \to 0}(AZ) =$ arc $\Delta K$. That is, when $\Delta Z$ is tangent to the spiral at $\Delta$, then $AZ =$ arc $\Delta K$, as claimed. Archimedes, of course, does not take "little $d\varphi$ slices." Instead, he first assumes $AZ >$ arc$\Delta K$ and gets a contradiction by making $d\varphi$ sufficiently small. Second he assumes $AZ <$ arc $\Delta K$ and again gets a contradiction by making $d\varphi$ sufficiently small. He then concludes equality.

**Note.** Archimedes addresses the area bounded by a spiral in his Proposition 24:

> "**Proposition XXIV.** The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle' [$= \frac{1}{3}\pi(2\pi a)^2$, where the spiral is $r = a\theta$]."

We give a proof of this in the supplement using Riemann integration and polar coordinates. Archimedes gives an argument very similar to this. Similar to his proof of Proposition 20, he first assumes the area bounded by the spiral is less than one-third of the 'first circle' and gets a contradiction by choosing a Riemann sum of inscribed sectors that are sufficiently small (see Figure 4.5 below). Second, he assumes the area bounded by the spiral is greater than one-third of the 'first circle' and gets a contradiction by choosing a Riemann sum of circumscribed sectors that are sufficiently small. He then concludes that equality holds. In fact, this is Archimedes' approach to many of his area and volume arguments; the approach is called "the method of exhaustion." He also uses this idea in approximating $\pi$. This is explained in the supplement to this section, Archimedes: 2,000 Year Ahead of His Time (in PowerPoint, with a transcript available in PDF).

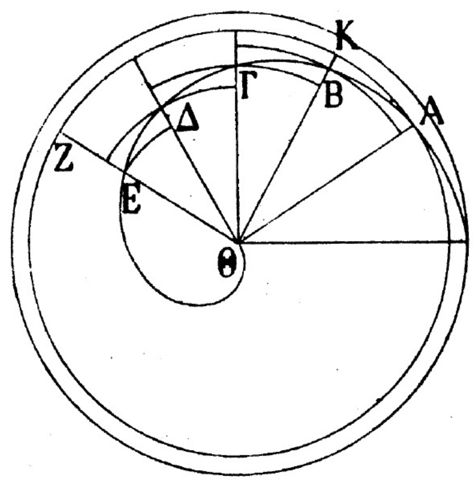



**Fig. 4.5.** Drawing by Archimedes for the area of the spiral (left)

*Revised: 4/20/2024*