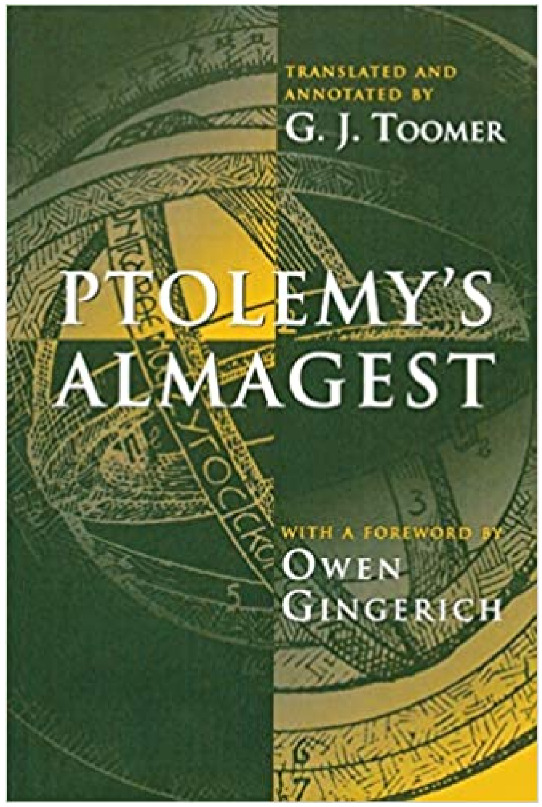
Chapter 5. Trigonometry

Section 5.1. Ptolemy and the Chord Function.

Note. Claudius Ptolemy (circa 85 CE–165 CE) was born, worked, and died in Egypt. He is best known for his work the *Almagest*, whose geocentric model of the universe dominated the area of astronomy for around 1400 years (until the acceptance of Copernicus heliocentric model, following the work of Kepler and Galileo). The *Almagest* includes observations Ptolemy made from Alexandria Egypt between the years 127 and 141 CE. We'll discuss some of the content of the *Almagest* below. Ptolemy also published *Handy Tables* which were improved, more accurate versions of many of the tables appearing in the *Almegest*. He wrote a "popular level" description of his ideas on planetary motion, *Planetary Hypothesis*. This consisted to two books, with simplification of the mathematical arguments and replacements of them in terms of mechanical models. In *Analemma* he described the construction of sundials based on projections onto the celestial sphere (that is, the night sky viewed as a sphere around the Earth on which the stars, sun, moon, and planets move). He considered mappings of the celestial sphere onto a plane in the form of stereographic projection in his *Planisphaerium* (for an explanation of stereographic projection in the setting of the complex plane, see my online notes for Complex Analysis 1 [MATH 5510] on Section I.6. The Extended Plane and Its Spherical Representation). His *Geography* consisted of eight books and attempted to give maps for the known world in terms of longitude and latitude; using the available data of the time, this work contained a number of inaccuracies. Ptolemy also wrote *Optics*, consisting of five books. It involved studies of color, reflection, refraction, and mirrors. This also contained a mathematical development of the subject as well as much physical data. This historical information (the photo next image, and much of the description of the *Almagest* given below) is from the MacTutor History of Mathematics Archive biography of Ptolemy (accessed 5/15/2023).

Note. The *Almagest* is in print in English as *Ptolemy's Almagest*, Revised Edited Edition, translated and annotated by G. J. Toomer, Princeton University Press (1998). According to the Foreward by Owen Gingerich, the *Almagest* we written in Greek in Egypt around 150 CE under the Greek title *Mathematike Syntaxis* ("Mathematical Treatise"). Ptolemy's work was based on an Earth centered (i.e., geocentric) model of the universe (previously advocated by Plato and Aristotle). His model would be the dominant one until Nicolaus Copernicus (February 19, 1473–May 24, 1543) published his heliocentric model in *On the Revolutions of he Heavenly Spheres* (*De revolutionibus orbium coelestium*) in Nuremberg in 1543 (though Aristarchus of Samos [circa 310 BCE–circa 230 BCE] advocated a heliocentric model).

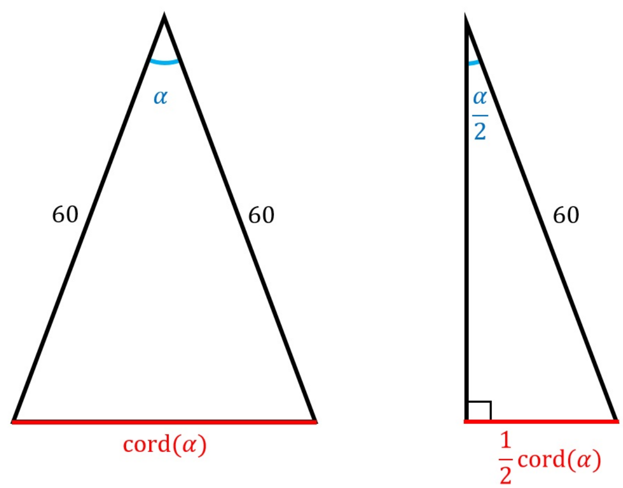


Note. The *Almagest* was translated into Arabic in the eighth or ninth centuries, with the title *Al-majisti* from which its current title comes. Knowledge of the work was lost to western Europe by the early middle ages (read this as around the fifth century with the collapse of Roman civilization), though it continued to circulate in the (eastern) Byzantine empire. Johann Regiomontaus (June 6, 1436–July 6, 1476) translated it into Latin and this version was the second scientific book to be printed (after Euclid's *Elements*). It was printed in 1496 according to Ostermann and Wanner, although Toomer in his translation reports that it was first printed in 1515 (in Venice).

The frontispiece of the first printed version of the *Almagest* depicting Ptolemy an the left and Regiomonataus on the right; from the Wikimedia.org webpage (accessed 5/15/2023)

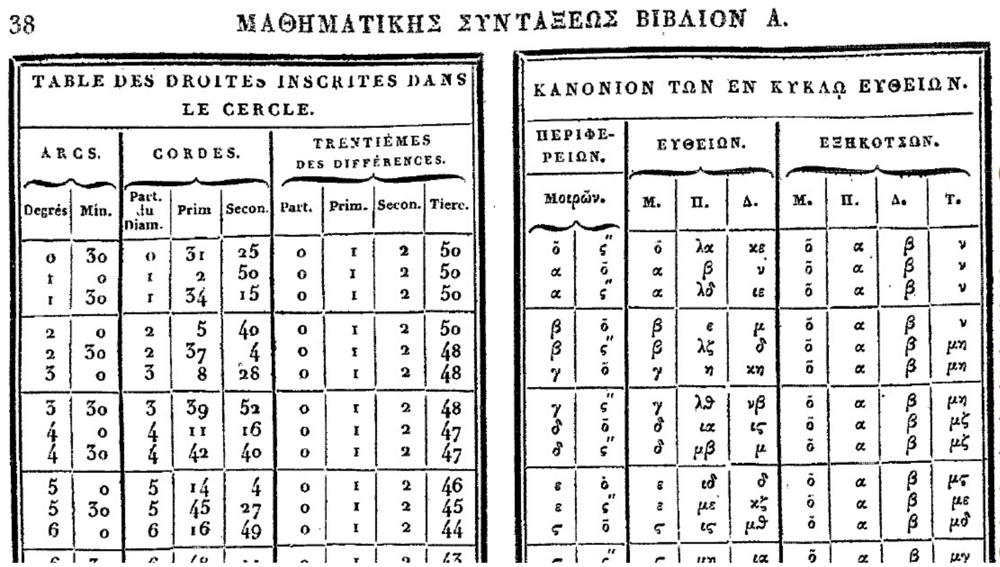
Note. The thirteen books of the *Almagest* describe how to calculate the positions of the five known planets at any time, past, present, or future. This was revolutionary in itself since it relied on mathematical computations to describe the workings of the physical world (regardless of the model used). This evolves into Johannes Kepler's (December 27, 1571–November 15, 1630) precise mathematical description of planetary motion in a heliocentric model, the explorations of physical principles of motion by Galileo Galilei (February 15, 1564–January 8, 1642), and ultimately the development of classical mechanics by Isaac Newton (January 4, 1643–March 31, 1727).

Note. Using geometry, Ptolemy set up models to give the positions of the sun, moon, and planets using motion along circles and epicyclces (smaller circles with their centers on the larger circles; the epicycles were used to explain the apparent retrograde motion of the planets). For more on epicycles, see my online notes for Astronomy (PS 215 at Auburn University) on Chapter 3. Early Astronomy); see Figure 3.12. To perform the necessary mathematical manipulations, Ptolemy introduces the *chord function* of an angle, cord (α). It measures the side of an isosceles triangle with equal sides of length 60 and the angle included by these equal sides of size α, as follows.



Based on the right triangle above, $\text{cord}(\alpha) = 2(60)\sin(\alpha/2) = 120\sin(\alpha/2)$. Ptolemy gives values of cord(α) of all α between $1/2^\circ$ and 180° taken at $1/2^\circ$ increments. In Figure 5.2, the angles are measured in degrees and minutes. The value of the chord function ("cordes") is recorded in the Babylonian sexagesimal system where the first decimal part, or *partes minutae primae* (first small parts), and the second decimal part, or *partes minutae secondae* (second small parts), are given in Figure 5.2. By the way, Ostermann and Wanner state (see page 113) that these decimal parts are where we get our words for minutes and seconds.

As an example, consider $\alpha = 6°$ (the last visible entry in Figure 5.2). We have $\text{cord}(6°) = 120\sin(3°) \approx 6.280315$. Ptolemy reports the result of 6 16 49. Since this is a sexigesimal representation, we interpret it as $6 + 16/60 + 49/60^2 \approx 6.280278$. Notice that $|6.280315 - 6.280278| \times 3600 = 0.1332 < 1$ so that Ptolemy's value is accurate to the nearest 1/3600 (or, if you like, to the nearest second).



38 ΜΑΘΗΜΑΤΙΚΗΣ ΣΥΝΤΑΞΕΩΣ ΒΙΒΛΙΟΝ Α.

TABLE DES DROITES INSCRITES DANS LE CERCLE.

ARCS.		CORDES.			TRENTIÈMES DES DIFFÉRENCES.			
Degrés	Min.	Part. du Diam.	Prim	Secon.	Part.	Prim.	Secon.	Tierc.
0	30	0	31	25	0	1	2	50
1	0	1	2	50	0	1	2	50
1	30	1	34	15	0	1	2	50
2	0	2	5	40	0	1	2	50
2	30	2	37	4	0	1	2	48
3	0	3	8	28	0	1	2	48
3	30	3	39	52	0	1	2	48
4	0	4	11	16	0	1	2	47
4	30	4	42	40	0	1	2	47
5	0	5	14	4	0	1	2	46
5	30	5	45	27	0	1	2	45
6	0	6	16	49	0	1	2	44

ΚΑΝΟΝΙΟΝ ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ.

ΠΕΡΙΦΕΡΕΙΩΝ.		ΕΥΘΕΙΩΝ.			ΕΞΗΚΟΣΤΩΝ.			
Μοιρῶν.		Μ.	Π.	Δ.	Μ.	Π.	Δ.	Τ.
ο̄	ς''	ο̄	λα	κε	ο̄	α	β	ν
α	ο̄	α	β	ν	ο̄	α	β	ν
α	ς''	α	λδ	ιε	ο̄	α	β	ν
β	ο̄	β	ε	μ	ο̄	α	β	ν
β	ς''	β	λζ	δ	ο̄	α	β	μη
γ	ο̄	γ	η	κη	ο̄	α	β	μη
γ	ς''	γ	λθ	νβ	ο̄	α	β	μη
δ	ο̄	δ	ια	ις	ο̄	α	β	μζ
δ	ς''	δ	μβ	μ	ο̄	α	β	μζ
ε	ο̄	ε	ιδ	δ	ο̄	α	β	μς
ε	ς''	ε	με	κζ	ο̄	α	β	με
ς	ο̄	ς	ις	μθ	ο̄	α	β	μδ

0	30	0	31	25
1	0	1	2	50
1	30	1	34	15
2	0	2	5	**39**
2	30	2	37	4
3	0	3	8	28
3	30	3	39	**53**
4	0	4	11	**17**
4	30	4	42	40
5	0	5	14	4
5	30	5	45	27
6	0	6	16	49

Fig. 5.2. Beginning of Ptolemy's chord table (left as published in Paris 1813); correct values (right)

Note. For the remainder of this section we **modify the chord function** to relate to an isosceles triangle the equal sides of length 1. **With this revised definition we have** $\text{cord}(\alpha) = 2\sin(\alpha/2)$. Since $\sin(30°) = 1/2$, then $\text{cord}(60°) = 1$; notice that $\alpha = 60°$ implies that we are dealing with a hexagon inscribed in a circle of radius 1 (see the hexagon entry Table 1.1 of Section 1.7. The Pythagorean Theorem). We also see from the last entry of Table 1.1 (the entry for the decagon) that if length of each side of a decagon is 1 then the radius of the circumcircle is Φ (the golden ratio). If we scale this by $1/\Phi$, then the circumcircle has radius 1 and the side of the decagon has length $1/\Phi$. Therefore, $\text{cord}(360°/10) = \text{cord}(36°) = 1/\Phi$. Just as Archimedes bisected angles to estimate the value of π (see my PowerPoint Supplement. Archimedes: 2,000 Year Ahead of His Time), Ptolemy can find $\text{cord}(\alpha/2)$ based on $\text{cord}(\alpha)$. He also finds the chord function of a sum or difference using the identity:

$$2\,\text{cord}(\alpha+\beta) = \text{cord}(\alpha)\,\text{cord}(180°-\beta)+\text{cord}(180°)\,\text{cord}(\beta) \quad (5.1)$$

Notice that, for example, Ptolemy can find $\text{cord}(3°)$ by (1) bisecting 60° to get $\text{cord}(30°)$, (2) using (5.1) for a difference to get $\text{cord}(36° - 30°) = \text{cord}(6°)$, then (3) bisecting 6° to get $\text{cord}(3°)$. Because of the failure to constructively trisect an angle, he cannot directly find $\text{cord}(1°)$ and must instead use interpolation to get the level of precision given in Figure 5.2. Ptolemy's proof of 5.1 is based on the following lemma.

Lemma 5.1 (Ptolemy's Theorem). Let a quadrilateral with sides a, b, c, d be inscribed in a circle. Then the diagonals δ_1 and δ_2 satisfy $ac + bd = \delta_1\delta_2$.

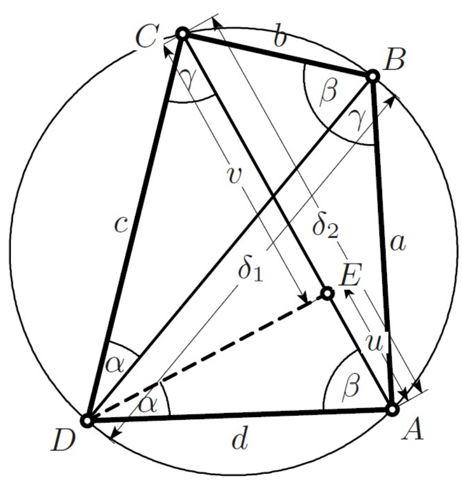


Fig. 5.4. Proof of Ptolemy's lemma

Note. Notice that Ptolemy's chord function can be used to estimate π. We can reproduce Archimedes inscribed 96-gon technique for estimating π (using a circle with radius 1) by taking $\alpha = (360/96)^\circ = 3.75^\circ$. We have $\text{cord}(3.75^\circ) = 2\sin((3.75/2)^\circ) = 2\sin(1.875^\circ) \approx 0.065438$ and, since the circumference of the circumcircle is 2π, $2\pi \approx 96\,\text{cord}(3.75^\circ) \approx 96(0.065438) = 6.282048$ so that $\pi \approx 3.141024$. Recall that Archimedes had the lower estimate of $3\,\frac{10}{71} \approx 3.140845$ (notice that the chord function gives a better lower bound, but we have used an approximation to $\sin(1.875^\circ)$ that is accurate to 6 decimal places, whereas Archimedes used several rational number bounds on trigonometric functions). Ptolemy used a circumscribed 360-gon to estimate π (this requires an estimate of $\sqrt{3}$, as was required by Archimedes when he was dealing with a circumscribed 96-gon). This gave Ptolemy an upper bound on π of $3\,\frac{17}{120} \approx 3.141667$. Using an inscribed 360-gon and an estimate of the chord function to six decimal places, a lower bound on π is $(360/2)\,\text{cord}(1^\circ) = 360\,\sin((1/2)^\circ) \approx 3.141553$. The first two books of the *Almagest* cover the cord function and measurements on the celestial sphere.

Note. We make a final comment about estimated π with the polygons. If we inscribe a regular n-gon in a circle of radius 1, then the central angle associated with a side of the n-gon is of size $\alpha = 2\pi/n$ in radians (here we treat radians simply as an arbitrary measure of angles, related to degrees by the conversion factor $180^\circ = \pi$ radian), then the perimeter of the n-gon is

$$n\,\text{cord}(\alpha) = n\,\text{cord}\left(\frac{2\pi}{n}\right) = n\left(2\sin\left(\frac{\alpha}{2}\right)\right) = n\left(2\sin\left(\frac{\pi}{n}\right)\right) = 2n\sin\left(\frac{\pi}{n}\right).$$

Now by L'Hôpital's Rule,

$$\lim_{n\to\infty} n\,\mathrm{cord}(\alpha) = \lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\lim_{n\to\infty}\frac{\sin(\pi/n)}{1/n}$$

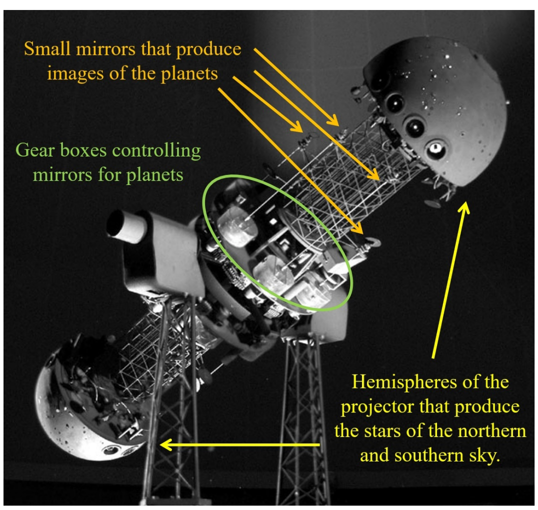
$$\stackrel{0/0}{=} 2\lim_{n\to\infty}\frac{\cos(\pi/n)[-\pi/n^2]}{-1/n^2} = 2\pi\lim_{n\to\infty}\cos\left(\frac{\pi}{n}\right) = 2\pi\cos\left(\lim_{n\to\infty}\frac{\pi}{n}\right) = 2\pi\cos(0) = 2\pi,$$

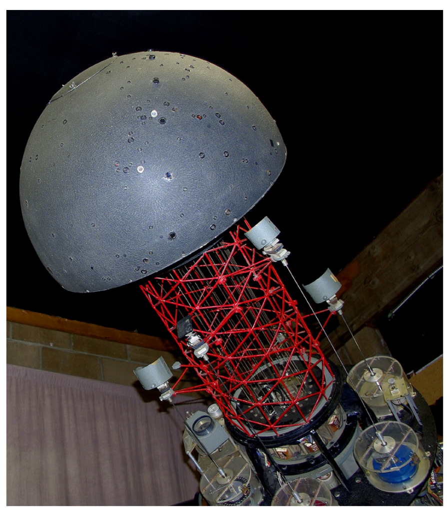
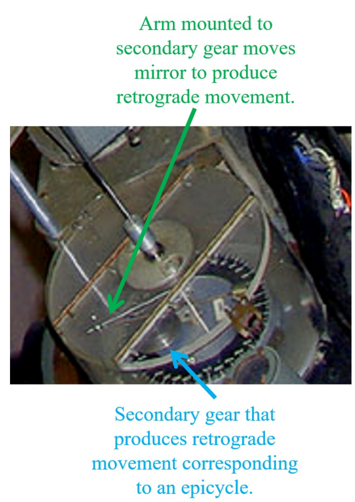
where we can take the limit since the cosine function because it is continuous. Of course this is artificial as a historical observation since since Ptolemy (and Archimedes) predate the formal concept of a limit (and L'Hôpital's Rule) by several hundred years. Also, we have chosen to represent angles in radians as an arbitrary unit, but it is necessary to measure angles in radian so that we can differentiate and apply L'Hôpital's Rule (the fact that the derivate of $\sin x$ is $\cos x$ only holds when x is measured in radians). None-the-less, this illustrates that the concept of approximation of the circle with regular n-gons (where n is "large") in finding approximations of π is a valid one.

Note. Book 3 of the *Almagest* concerns the sun. Based on observations of solstices and equinoxes (the dates on the calendar marking the changes of season), Ptolemy proposed a model for the sun involving uniform circular motion around the Earth, but with the Earth slightly off center from the orbit. Books 4 and 5 cover the motion of the moon. Building on work of Hipparchus (190 BCE–120 BCE), Ptolemy considers three different periods of lunar motion and he improves on Hipparchus' model of this motion. In Book 6 Ptolemy presents a theory of lunar and solar eclipses. Books 7 and 8 cover the "fixed" stars. Ptolemy maintains that the stars are fixed, based on the locations he observes versus those recorded by Hipparchus about 200 years earlier. Books 7 and 8 also catalogue over one thousand stars.

Books 9 through 13 cover planetary motion. Before the *Almagest* there was no theoretical model explaining the complicated movements of the planets, so *this* is Ptolemy's main contribution and this is what he is primarily remembered for. His predictions matched the observable data and made accurate predictions. In Ptolemy's model, the path of a planet consists of uniform circular motion on an epicycle whose center moved on a primary circle. The primary circle has a center offset from the Earth and the epicycle moves around the primary circle uniformly NOT with respect to the center of the primary circle, but with respect to a point called the *equant* which is symmetrically placed on the opposite side of the center from the Earth. With this involved model, Ptolemy was able to make accurate predictions of the locations of the planets in the future.

Note. We conclude this section with an application of Ptolemy's model. A planetarium projector reproduces the night sky as viewed from the Earth. So an older project (one from before everything was generated digitally) had to mechanically reproduce the movement of the sun, moon, and planets relative to the fixed starts. The stars and planets (and the sun and moon) are projected onto the dome of the planetarium. The fixed stars themselves are produced by two hemispheres with holes drilled in them and containing a light source. The planets are projected onto the dome by individual light sources which are bounced off of a little mirror which can be rotated to simulate motion of the planet over time. If the little mirrors simply rotated on a gear, then the planet would just move around the dome without reproducing the retrograde motion. So in order to produce the retrograde motion, the little mirrors are mounted on an arm whose motion is affected be a secondary gear which reproduces the retrograde motion and acts as Ptolemy's epicycles. The ETSU Planetarium began operation in fall of 1962. It had a Spitz Model A3P Projector (state of the art technology, for the time). The projector produced images of 6,000 stars on the 24 foot dome. The planetarium is still in use (though the old Spitz projector was replaces in the 2000s with a digital projector) and is located in Room 207 of Hutcheson Hall. Free public shows are still given monthly and details are on the ETSU Planetarium webpage (accessed 5/16/2023). The following images are from the Planetarium Projector Museum, Spitz Space Transit Planetarium Projector webpage. They show the structure of a Spitz projector and give a close-up view of the gears that generate the images of the planets on the planetarium dome. The secondary gear and the arm mounted to it that produces the retrograde motion can be seen on the right. All this is based on Ptolemy's model because it is so good at producing the observed night sky!



Images from the Planetarium Projector Museum, Spitz Space Transit Planetarium Projector webpage (accessed 5/16/2023)

Revised: 5/16/2023