Section 5.10 Supplement. Feynman's Lost Lecture

Note. We now turn our attention a geometric argument (in contrast to an argument based on calculus) for the claim that an inverse square law of attraction results in orbits that are in the shapes of conic sections. This is stated in the University of California Press 1962 version of *Principia* translated by Motte and revised by Cajori as:

> **Corollary 1.** From the three last Propositions [11, 12, and 13] it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR, and at the same time is urged by the action of a centripetal force that is inversely proportional to the square of the distance of the places from the centre, the body will move in one of the conic sections, having the focus in the centre of force; and conversely." (See page 61.)

Note. On March 13, 1964 famed physicist Richard Feynman gave a presentation at the California Institute of Technology on "The Motion of Planets Around the Sun." This has become known as "Feynman's Lost Lecture, The Motion of Planets Around the Sun," and appears in David Goodstein and Judith Goodstein's *Feynman's Lost Lecture, The Motion of Planets Around the Sun* (NY: W.W. Norton & Company, 1996). The audio of the lecture, which runs 1:17:44, is on YouTube. In the Goodstein's book, it is also revealed that James Clerk Maxwell (of electromagnetism fame) in 1877 published the same proof as that given by Feynman (see Ostermann and Wanner, page 149).

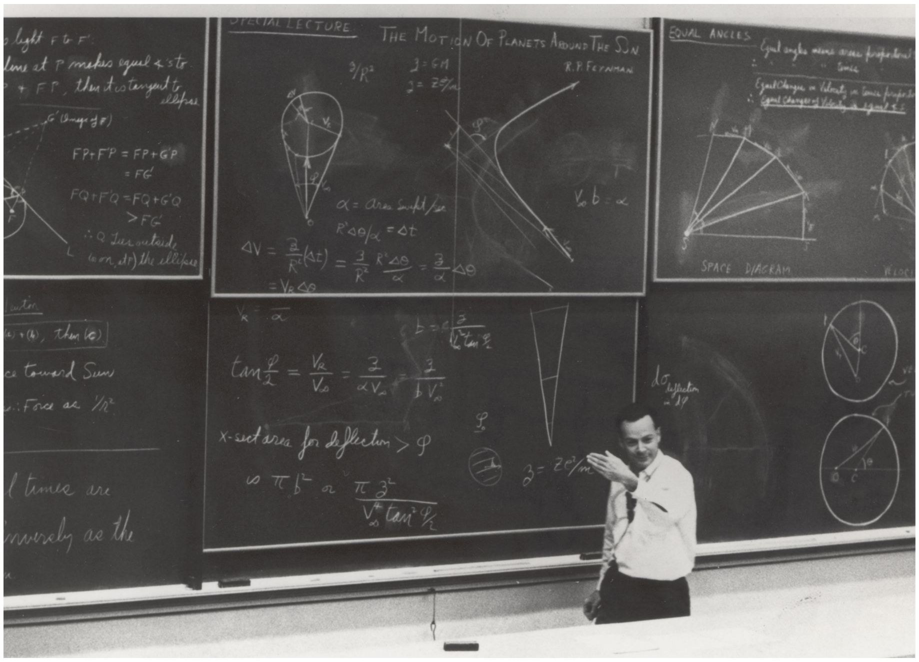


Image from Wikipedia page on "Feynman's Lost Lecture"

Note. We start with the assumption of an inverse square law of attraction. As Newton does above, we allow the force to act as an impulse, followed by a small interval of uniform motion, and the another impulse, etc. However, this time we follow the body in motion through a fixed angle $\Delta\varphi$ between the impulses, as shown in Figure 5.32.

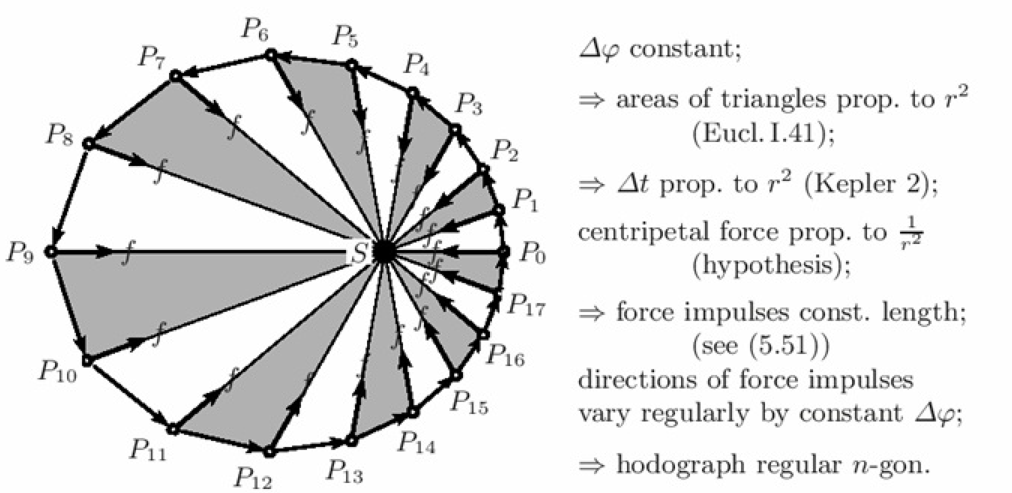
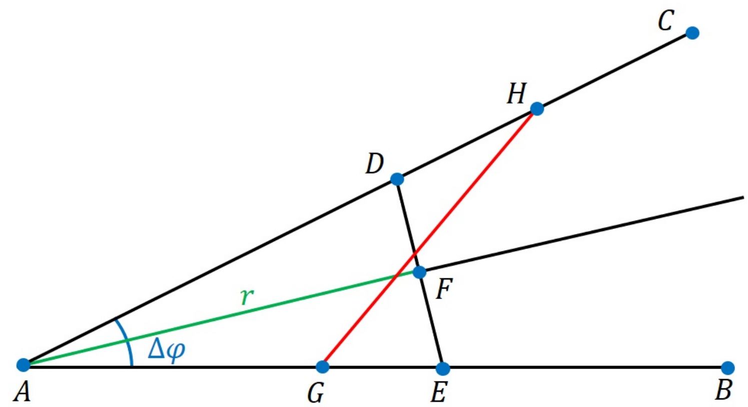


Fig. 5.32. Feynman's variant with equal angles instead of equal time steps

We now argue that the area swept out as the body moves through angle $\Delta\varphi$ is proportional to the distance squared r^2. We give an argument based on Brian Beckman's "Feynman Says: Newton implies Kepler, No Calculus Needed!," *The Journal of Symbolic Geometry*, **1**, 57–72 (2006) (available online on Andrew Long's Northern Kentucky University webpage; accessed 10/16/2021).



Based on Figure 7 from Beckman's paper

In the continuous case, the distance r is a function of time and/or the eccentric anomaly angle. The problem with the discrete case is that over a time interval Δt or an angle $\Delta\varphi$, the distance r is not a constant. We want to establish proportional relationships with r^2 and $1/r^2$, so we need a way to deal with the variable nature of r; this does not seem to be explicitly done in Feynman's lecture nor in the explanation given in Ostermann and Wanner. Beckman gives a definition of the distance r from the sun to the body over angle $\Delta\varphi$ using his Figure 7 (see above). Suppose the Sun is located at point A. Over angle $\Delta\varphi$, suppose the path of the body in the discrete model is given by the red line segment GH. Bisect angle $\Delta\varphi$ with a ray through point A. We consider a segment DE perpendicular to this ray such that the area of triangle ADE is the same as the area of triangle

AHG (in Figure 7, when segment DE is such that points E and G coincide the area of triangle ADE is smaller than the area of triangle AHG, and when points D and H coincide the area of triangle ADE is larger than the area of triangle AHG; so continuity and the Intermediate Value Theorem are playing a role here). Label as point F the intersection of the bisecting ray through point A and the line segment DE. Define the distance r of the body from the sun over angle $\Delta\varphi$ as the distance from A to F. The height of triangle ADE is r and the base is $2r\sin\Delta\varphi$, so that the area of triangle ADE is $2r^2\sin\Delta\varphi$. Hence the area of triangle ADE (and so also the area of triangle AHG) is proportional to r^2. By Kepler's Second Law, the planets sweep out equal areas in equal times so that the time intervals associated with various $\Delta\varphi$ are also proportional to r^2. Consider triangle SP_iP_{i+1} for some i (see Figure 5.32 above). Then the area of this triangle is proportional r^2 (where r^2 as as defined by Beckman) and time it takes the body to move from P_i to P_{i+1} is denoted Δt_i. So Δt_i is also proportional to r^2. Now, as described in the proof of Theorem 5.8 (Newton's Theorem 1 of *Principia Mathematica*), the impulse force applied to the body at point P_i is $f \cdot \Delta t_i$. Since we are hypothesizing an inverse square law of attraction here, then f is proportional to $1/r^2$ and hence the impulse force $f \cdot \Delta t_i$ is of constant magnitude (this is illustrated in Figure 5.32 by the constant length of each of the impulse force vectors). [However, the r defined in Beckman is not the r where the impulse is applied!!! Beckman's r is based on an angle $(\Delta\varphi)/2$ greater than the angle where the impulse is applied. So the $1/r^2$ proportionality is only approximate, and...???] In addition, the impulse force vectors form a "regular star" in that when all the impulse force vectors are drawn with their tails at point S of Figure 5.32, they are regularly spaced out with angles of $\Delta\varphi$ between consecutive impulse force vectors. We therefore have:

$$\begin{gathered}\textit{The force impulses will all have the same length.}\\ \text{Moreover, } \textit{their directions form a regular star.}\end{gathered} \tag{5.57}$$

Revised: 12/2/2021