Section 5.10. The Great Discoveries of Kepler and Newton

Note. In this section, we step aside from mathematics (arguably) and consider a solution to a problem from astronomy that was the topic of quantitative study for at least 2,000 years. We describe the motion of the planets. This is accomplished *empirically* with Kepler's Three Laws of Planetary Motion. Kepler, however, does not give an explanation of *why* the planets move as he describes, but simply gives the description of the movement. We derive Newton's inverse square law of gravity assuming Kepler's first and second laws (in Theorem 5.9; the argument is based on geometry as Newton originally presented it, and does not rely on calculus). Conversely, we show that Newton's inverse square law of gravity implies Kepler's First Law (for this, we give a geometric argument of Richard Feynman and discuss the history of this problem). Hence it is Newton's work which gives the *causal* explanation of the movements of the planets about the sun. The method by which Newton's gravitational "force" is transmitted remains mysterious until Einstein explains it in terms of curvature of spacetime in his General Theory of Relativity.

Note. Ostermann and Wanner claim that the three great works making the emergence of "modern science" (well, modern *physical* science, maybe) are:

1. Johannes Kepler (December 27, 1571–November 15, 1630), *Astronomia Nova* (1609); the full title in English is *New Astronomy, Based upon Causes, or Celestial Physics, Treated by Means of Commentaries on the Motions of the Star Mars, from the Observations of Tycho Brahe, Gent.* This includes Kepler's first two laws of planetary motion (his third law appears in his 1619

Harmonies of the World). Based on the precise (but naked eye) observations of Tycho Brahe (December 14, 1546–October 24, 1601), Kepler presented an argument that Mars has an elliptical orbit. The evidence of the elliptical shape of the orbits of the other planets was presented later.

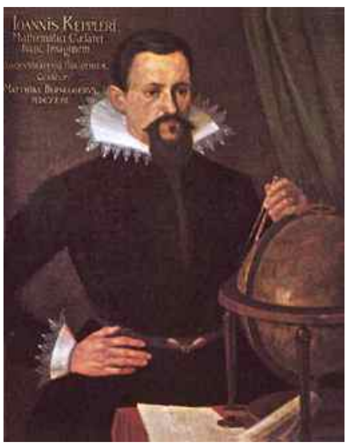
2. Galileo Galilei (February 15, 1564–January 8, 1642) *Discorsi e Dimostrazioni Matematiche* (1638); the full title in English is *Discourses and Mathematical Demonstrations Relating to Two New Sciences.* This is Galileo's final book and is written in the style of his *Dialogue Concerning the Two Chief World Systems* which caused him so much trouble with the church and the Roman inquisition. The book centers around conversations between three characters, Simplicio, Sagredo, and Salviati. This is the work where Galileo describes the motion of objects in free fall. He used inclined planes to slow the fall of objects and he deduced the quantitative behavior of an accelerating object (an impressive accomplishment, given that he did not have access to very accurate time pieces). From this work, it follows that objects in motion near the surface of the Earth follow parabolic trajectories.

3. Isaac Newton (December 25, 1642/January 4, 1643–March 20, 1727/Math 31, 1727) *PhilosophiæNaturalis Principia Mathematica* (1687); the full title in English is *Mathematical Principles of Natural Philosophy.* In this monumental work, Newton presents his Universal Law of Gravitation and his Laws of Motion. He used these results to derive Kepler's Laws of Planetary Motion. Surprisingly, he gives geometric arguments for his results instead of using the calculus techniques which are used to demonstrate these ideas in a physics class today (and he is the first to develop these techniques!). An 1846 copy of

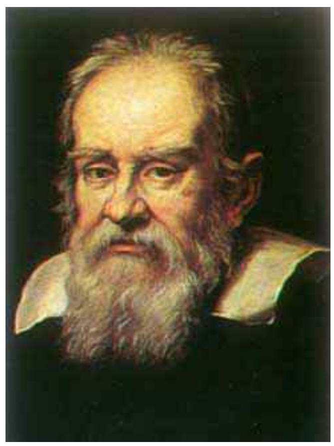
Newton's *Principia* translated by Andrew Motte is available on the Red Light Robber website (accessed 10/5/2021).

This historical information is based on the Wikipedia pages for Kepler, Galileo and Newton, where reliable references for these claims are given (accessed 9/24/2021).

Note. Each of these books is still in print. The following images of Kepler, Galileo, and Newton are from the MacTutor History of Mathematics Archive webpage. The images of the books are from Amazon.com.

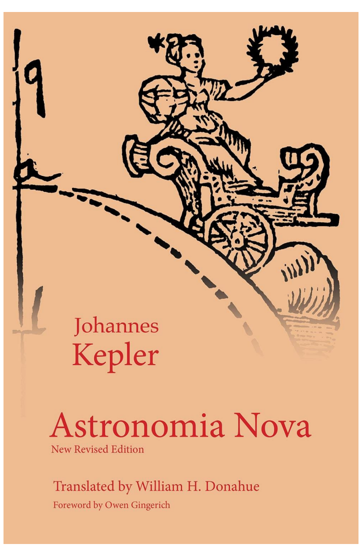


Johannes Kepler

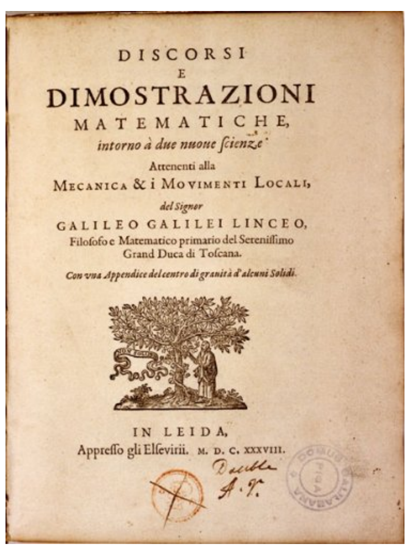


Galileo Galilei

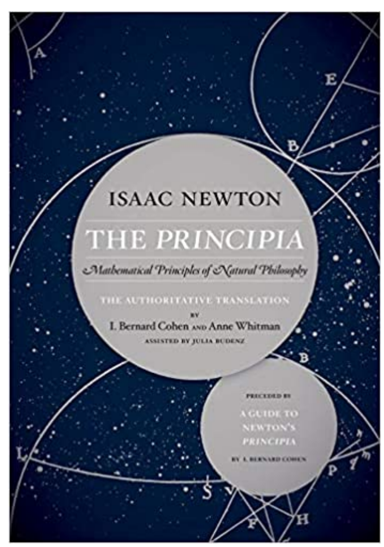
Issac Newton



Astronomia Nova



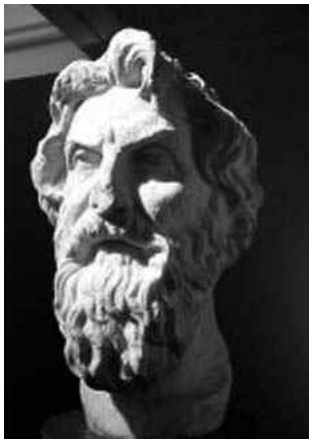
Discorsi



Principia Mathematica

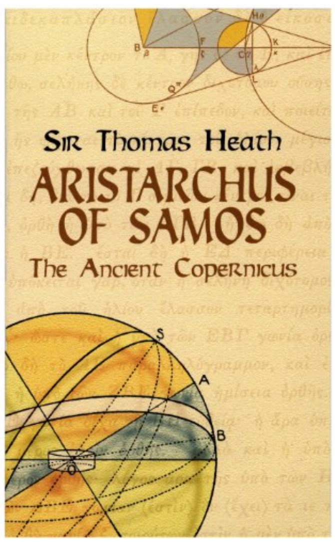
Note. Two other people are relevant to this story. Aristarchus of Samos (circa 310 BCE–circa 230 BCE) was the first person in history known to have proposed that the Earth goes around the sun ("heliocentrism") in a circular orbit with the Sun at the center. Aristarchus also proposed that the stars were themselves suns, just very far away. The original work in which he presented these ideas is lost, but it is known due to a reference of Archimedes in his *Sand Reckoner*. Some of Aristarchus' work survives in the form of *On the Sizes and Distances of the Sun and Moon*, which is still in print and appears in Thomas Heath's *Aristarchus of Samos: The Ancient Copernicus*. Nicolaus Copernicus (February 19, 1473–May 24, 1543), a Polish astronomer, published *De Revolutionibus Orbium Coelestium* (In English, *On the Revolutions of the Celestial Spheres*) in 1543. He completed the work around 1532, in which he proposed heliocentrism with the planets orbiting the sun in circular orbits. It seems that Copernicus was aware of Aristarchus' work, and that of others, based on early versions of the *De Revolutionibus*, but he dropped these references in the 1543 version. Out of fear of of backlash from the church, he delayed publication until shortly before his death. The impact was so large, that the would "revolution" in the tile (previously just having the meaning in the sense of "the Earth revolves around the sun") took on the meaning of a major shift in thought (as in "revolutionary"). Copernicus book was widely circulated, but it has the reputation of not being widely read, as indicated by the title of Owen Gingerich's 2004 *The Book Nobody Read*. This historical information is based on the Wikipedia pages for Aristarchus and Copernicus, where reliable references for these claims are given (accessed 9/24/2021). The following images of Airstarchus and Copernicus are from the MacTutor History of Mathematics Archive webpage.

The images of the books are from Amazon.com.

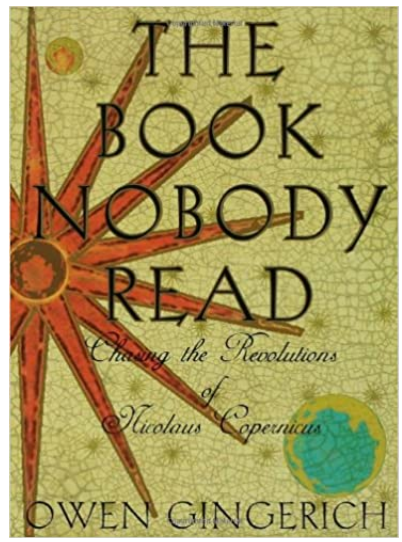


Aristarchus of Samos

Nicolaus Copernicus



Aristarchus: The Ancient Copernicus



The Book Nobody Read

Note. At the turn of the 17th century (i.e., around the year 1600), it was thought that the planets (well, Mercury through Saturn, since Uranus and Neptune were not known at the time) revolved around the Sun on "eccentric circles"; that is, on circles with their center near the Sun. This met the observational data (which was largely naked-eye data at the time), with the exception of Mars (this is due to the fact that the eccentricity of Mars' orbit is the largest of the eight planets, with the exception of Mercury which is particularly difficult to observe, given its nearness to the Sun). By careful study of Tycho Brahe's data on Mars, Kepler was able to determine that Mars orbits the Sun in an ellipse with the Sun at one focus. Kepler's Three Laws of Planetary Motion are:

Kepler's 1st Law. Planets move on elliptic orbits with the Sun at one of the foci.

Kepler's 2nd Law. The planets orbiting the Sun sweep out equal areas in equal time.

Kepler's 3rd Law. The squares of the periods of revolution are proportional to the cubes of the semi-major axes.

A nice climax of the Calculus class sequence is that Kepler's laws can be proved using the techniques developed in calculus and by assuming Newton's Universal Law of Gravitation. For these derivations, see my online Calculus 3 (MATH 2110) on 13.6. Velocity and Acceleration in Polar Coordinates. These notes also include some history. For additional historical comments, see my online notes for freshman Astronomy (these notes are based an a class at Auburn University, PS 215, which I taught in summer 1990) on Chapter 4. The Renaissance and Chapter 5. Isaac Newton and the Laws of Motion

Note. Before we discuss Kepler's successes, we mention one of his other ideas. . . There were only six known planets in Kepler's time. It was also known that there were only regular "platonic" solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron (see 2.5. Book XI. Spatial Geometry and Solids, especially Figure 2.27). Kepler thought that there was a connection between the number of the number of planets and number of platonic solids. Carl Sagan describes Kepler's original idea in his PBS series *Cosmos*, Chapter 3. The Harmony of the Universe as

> "In the course of a lecture on astrology, Kepler inscribed within the circle of the Zodiac a triangle with three equal sides. He then noticed, quite by accident, that a smaller circle inscribed within the triangle bore the same relationship to the outer circle as did the orbit of Jupiter to the orbit of Saturn. Could a similar geometry relate the orbits of the other planets?" (33:09–33:36)

By nesting (or inscribing) the five platonic solids in one another, the distances of the planets from the Sun would then be determined. The solids were inscribed in one another (from the inside out) as octahedron, isocsahedraon, dodecahedron, tetrahedron, and cube; the outer three solids can be seen in Figure 5.19 (see The Oracles's webpage on "Mysterium Cosmographicum by Johannes Kepler" for a clearer view; accessed 9/25/2021). In ancient Greek astronomy, the planets were thought to move on crystalline spheres. In this same spirit, Kepler considers spheres that are inscribed in or circumscribed around the five platonic solids (this is how *six* orbits are determined by *five* solids).

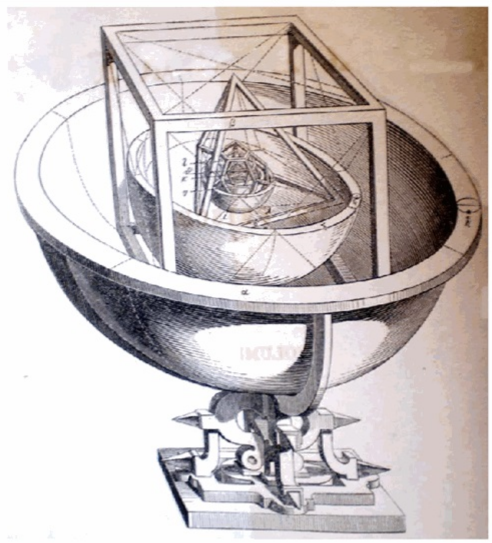
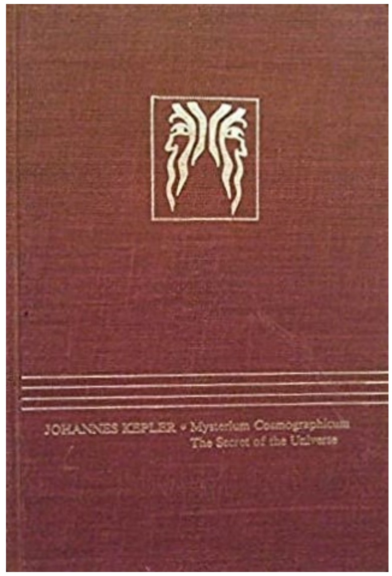


Fig. 5.19. Kepler's cosmological model

Quoting form Carl Sagan's *Cosmos*, NY: Random House (1980):

> "In these perfect forms, he believed he had recognized the invisible supporting structures for the sphere of the six planets. He called his revelation The Cosmic Mystery. The connection between the solids of Pythagoras and the disposition of the planets could admit but one explanation: the Hand of God, Geometer." (See page 57.)

Kepler published his ideas in 1596 in *Mysterium Cosmographicum* (in English, *The Secret of the Universe*). But Kepler's model did not agree with the observed data. In addition, the discovery of additional planets removes the special numerical nature of Kepler's model. One could argue that Kepler's approach was almost numerological in nature. In his defense, mathematical models of the positions of the planets (even in the case of geocentric models) had been somewhat successful. He was just looking for the wrong kind of mathematical model; remember, Kepler predates anything we would recognize as physical laws expressed in mathematical terms (this would start with Galileo, as mentioned above).



Mysterium Cosmographicum, Eastoan (1981)

Note. We now discuss how, based on data for Mars, Kepler discovered his first law. In Chapter 56 of *Astronomia Nova*, Kepler presents the image in Figure 5.28 (left). A diagram of Kepler's figure is given in Figure 5.28 (right). The solid circle is the best eccentric circle for the orbit of Mars. For scale, we take the radius of this circle to be 1 unit, let the center be O, and suppose that the circle rotates about point S. At one point in time Mars is at its greatest distance from the vertical axis SOC. If B is the point on the circular orbit at this point in time, then the distance from Mars to the vertical axis is 1. But based on the data of Mars as measured by Tycho Brahe, the actual position of Mars at this time was at point B' which is at a distance of $1/1.00429 \approx 0.99573$. Kepler recognizes this value as $\cos 5°18'$ (or, equivalently, recognizes 1.00429 as $\sec 5°18'$). The text book quotes him as saying:

> "When my triumph over Mars appeared to be futile, I fell by chance on the observation that the secant angle 5°18′ is 1.00429, which was the error of the measure of the maximal point. I awoke as if from sleep, & a new light broke on me." (*Astronomia Nova*, 1609, Cap. LVI, page 267)

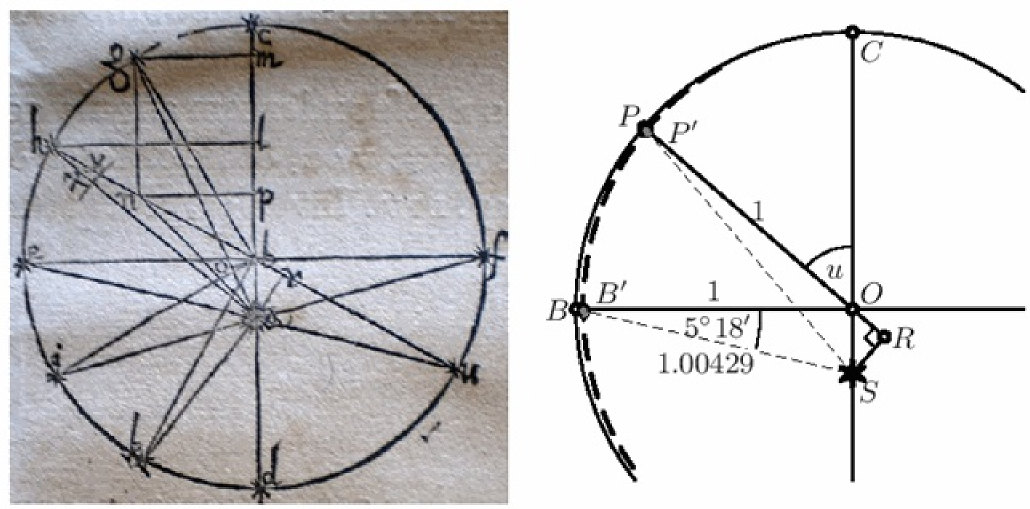
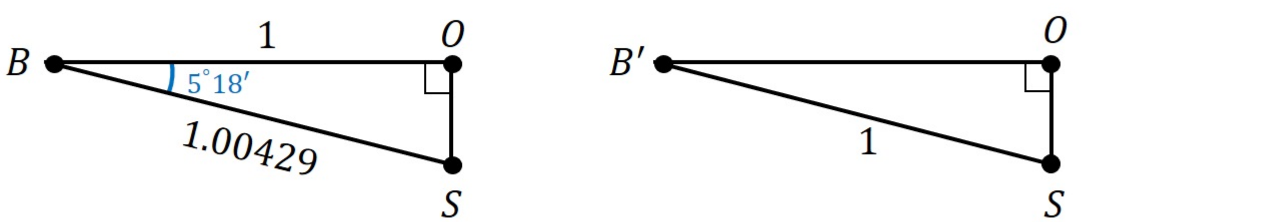


Fig. 5.28. The discovery of Kepler's first law (*Astronomia Nova*, Chap. 56); Kepler's drawing (left), modern drawing (right).

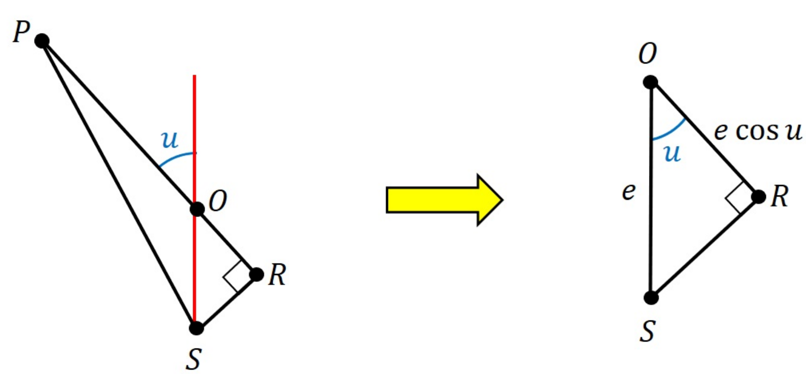
So he replaces the point B with point B' by translating point B along segment BO to point B' in such a way that the distance from B' to S is 1. That is, he creates a new right triangle $B'OS$ with the hypotenuse $B'S$ of length equal to the length of the leg BO (and the other leg in both triangles is the segment OS). He makes similar changes in the other points on the solid circle (explained below), creating the dashed path in Figure 5.28 (left).



Similarly, for any point P on the solid circle, we form a right triangle with hypotenuse PS and one of the legs as a segment containing points P and O (say the endpoint of this leg is at point R and point R is the vertex of the right angle; see Figure 5.28 right). We again slide point P along the radial segment PO until the length of segment $P'S$ equals the length of leg PR. To quantify this, we need to find the length of segment OR. We introduce the angle u as shown in Figure 5.28 (right), form which we can deduce that the length of segment OR is $e\cos u$ (see the figure below). So the length of segment $P'S$ equals the length of segment PR equals $1 + e\cos u$:

$$P'S = PR = 1 + e\cos u.$$

The resulting distances very closely match those observed by Brahe. This is the form of an ellipse where u is the eccentric anomaly; see Theorem 5.9.A in Section 5.9. Trigonometric Formulas for Conics. However, to use Theorem 5.9.A we must rotate Figure 5.28 (right) 90° counterclockwise to put point S on the positive x-axis so that it is at the focus F, as in Figure 5.26 (left). Then the parameter u of Figure 5.28 (right) is 180° greater than the value of u as given in Figure 5.26 (left) and in Theorem 5.9.A. So with $a = 1$ and in terms of u as used in this section, we have that $r = a - ae\cos u$ implies $r = (1) - (1)e\cos(u + 180°) = 1 + e\cos u$, as needed.



Note. Newton's first two laws of motion, as stated in the text book, are the following:

Lex 1. Without force a body remains in uniform motion on a straight line.

Lex 2. The change of motion is proportional to the motive force impressed.

A somewhat more familiar statement of these are the following, from NASA's Glenn Research Center's webpage:

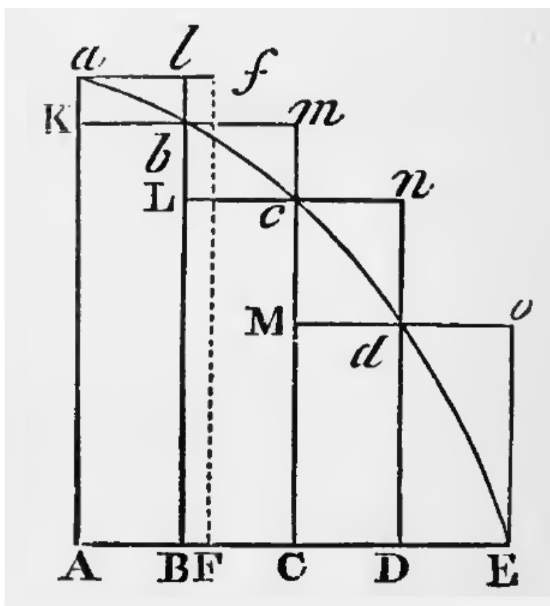
Newton's First Law. An object at rest remains at rest, and an object in motion remains in motion at constant speed and in a straight line unless acted on by an unbalanced force.

Newton's Second Law. The acceleration of an object depends on the mass of the object and the amount of force applied.

Newton's Second Law is quantified in the concise equation $F = ma$. We now give a geometric argument that Kepler's Second Law holds. We present the proof given by Newton in *Principia Mathematica* as Theorem 1. But we also need a result of Newton's which, in essence, allows him to take a limit. This result is his Lemma III in Book One of *Principia*. We now state it, along with his Lemma II. In Lemma II Newton introduces an idea very similar to Riemann's idea of Riemann sums as encountered in Calculus 1 when dealing with a regular partition (see my online Calculus 1 [MATH 1910] notes on Section 5.2. Sigma Notation and Limits of Finite Sums).

Lemma II. (From Newton's *Principia Mathematica*.)

If in any figure $AacE$, terminated by the right lines Aa, AE, and the curve acE, there be inscribed any number of parallelograms Ab, Bc, Cd, &c., and the sides, Bb, Cc, Dd, &c., comprehended under equal bases AB, BC, CD, &c, and the sides Bb, Cc, Dd, &c., parallel to one side Aa of the figure; and the parallelograms $aKbl$, $bLcm$, $cMdn$, &c., are completed: then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented *in infinitum*, I say, that the ultimate ratios which the inscribed figure $AKbLcMdD$, the circumscribed figure $AalbmcndoE$, and curvilinear figure $AabcdE$, will have to one another, are ratios of equality.



The Figure for Newton's Lemmas II and III from

Motte's 1846 translation of *Principia*.

Note. Lemma II has the requirement that the parallelograms have "equal bases" implies that we are dealing with a regular partition of the base AE. In Lemma III, Newton deals with a similar situation, but without the condition of equal bases.

Lemma III. (From Newton's *Principia Mathematica.*)
The same ultimate ratios are also ratios of equality, when the breadths AR, BC, DC, &c., of the parallelograms are unequal, and are all diminished *in infinitum*.

Note. Newton is taking some mathematical liberties here. He is depending on the drawing given above and making an assumption of convergence and continuity. The rigorous development of the ideas was given in the 1800s by Augustin Cauchy and Richard Dedekind. Euclid himself makes similar "continuity consideration" assumptions in his *Elements*; see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 2.2. A Brief Critique of Euclid.

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica.*)
"The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described." (This is just Kepler's Second Law.)

Note. In Calculus 3 (MATH 2110) a proof of Theorem 5.8 is given using vector values functions and cross products (see "Lemma" and Kepler's Second Law of Planetary Motion in my online notes on Section 13.6. Velocity and Acceleration in Polar Coordinates).

Note. In a physics or calculus class (see my online notes for Calculus 3 [MATH 2110] on Section 13.6. Velocity and Acceleration in Polar Coordinates, for example) you will encounter Newton's Law of Gravitation:

> If $\mathbf{r}$ is the position vector of an object of mass m and a second mass of size M is at the origin of the coordinate system, then a (gravitational) force is exerted on mass m of
>
> $$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}.$$
>
> The constant G is called the *universal gravitational constant* and (in terms of kilograms, Newtons, and meters) is 6.6726^{-11} $\mathrm{Nm}^2\mathrm{kg}^{-2}$.

We will give a proof of this based on Kepler's First and Second Laws. This proof appears in Newton's *Principia Mathematica* as Proposition 11. The statement given by Ostermann and Wanner is as follows; a proof will follow below.

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*.)
A body P, orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre S, satisfying the law

$$f = \frac{Constant}{r^2}, \text{ where } r \text{ is the distance } SP.$$

Note. We prove Theorem 5.9 below, but first we need an observation and a lemma. Figure 5.30 left (given below) reproduces an image from a 1684 manuscript of Newton's ("On the Motion of Bodies in an Orbit," in Newton's *Mathematical Papers*, volume VI, pages 30–91). The figure represents the path of a body attracted to "centre of force situated far away in the direction AC." The assumption that the attracting body is far away means that we assume the force it exerts to be always in the same direction. If the force were absent then the body would have a velocity tangent to the curved path in the *direction* AB (the magnitude of this initial velocity is not meant to be reflected by the *length* of AB). However, the force will cause the moving body to follow a curved orbit along the path AD over time interval Δt. If the body had no initial velocity at point A then it would move to point C over time interval Δt. Now AC and BD are parallel by the "far away" assumption/approximation. Since the force is assumed to have a constant magnitude and direction, then distance the object moves in the direction AC while traveling along curved path AD will equal the distance AC. So if we move away from point D in the direction away from the attracting body and introduce point B as shown, then we have $AC = BD$. Therefore $ACBD$ is a parallelogram. We know by Newton's Second Law ("Lex 2") that the force if proportional to distance AC. Since $AC = BD$ then then the force is proportional to distance BD:

> The acting force is proportional to the distance BD between the point on the tangent and the point on the orbit. (5.53)

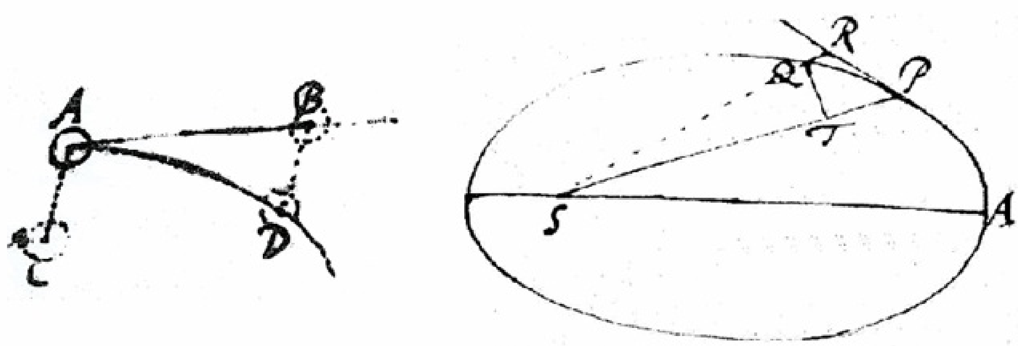


Fig. 5.30. Reproductions from Newton's autograph (1684), manuscript Cambridge Univ. Lib. Add. 3965[6]; the force acting on a moving body (left); picture for Newton's lemma (right). Reproduced by kind permission of the Syndics of Cambridge University Library

Newton's Lemma. Let APQ be an ellipse with focus S and suppose P to be the position of the planet moving towards Q, while the point R moves on the tangent with S, Q, P collinear. Let T be the orthogonal projection of Q onto PS (see Figure 5.30, right). Then if the distance PQ tends to zero, we have $RQ \approx (\text{Constant}) \cdot QT^2$, where the constant is independent of the position of P on the ellipse.

Note. We are now ready to prove Theorem 5.9 of Newton.

Note. We now take an in-depth look at the history of the converse. The information in this Note comes from Jason Socrates Bardi's The Calculus Wars: Newton, Lebniz, and The reatest Mathematical Clash of All Time (NY: Thunder's Mouth Press, 2006); see pages 118–121. Robert Hooke (July 18, 1635–March 3, 1703) took an interest in the gravitational nature of planetary motion and he wrote to Newton about this in 1679 and 1680. In 1684 Edmond Halley (November 8/October 28, 1656–January 25, 1742; famous for his prediction of the return in 1758 of the comet now named for him) met with Hooke and Christopher Wren (October 30/October 20, 1632–March 8, 1723). Halley was interested in the path of "his" comet, which had recently made an appearance in the inner solar system in 1682. Halley asked Hooke and Wren about the physical law that would determine the path, and Hooke correctly answered that the path would be determined by an inverse square law of attraction. Wren challenged Hooke to give a mathematical proof of his claim, but Hooke declined the challenge. Wren told Halley about Newton. Halley traveled to Cambridge and Trinity College and asked the same question of Newton. Newton answered immediately that orbiting objects are under an inverse law of attraction and the paths they follow are ellipses. Halley was pleased with the answer, since it matched what Hooke had claimed. Newton stated that he had calculated it years before but not have the calculation at hand. Halley returned to London and Newton later sent Halley two proofs, along with a copy of "On the Motion of Bodies in an Orbit" (which is mentioned above in connection with Figure 5.30). Halley was impressed and encouraged Newton to write more. Quoting from Bardi's book:

> "While some may think that Halley's greatest contribution was predicting the return of the comet he ultimately gave his name to, one could argue that in fact his greatest accomplishment was to convince Newton to publish one of the greatest books ever written—the *Principia*. In fact, Halley did not only cajole Newton into writing the *Principia*, he also oversaw the production of the book and personally underwrite the expense of publishing it in 1687, since the Royal Society could not scrape together the funds to do so." (See page 121.)

Note. We now discuss some controversy over the validity of the claim that Newton gives a complete proof that an inverse square law of attraction yields orbits in the shape of conic sections. We reference Robert Weinstock's "Newton's *Principia* and Inverse-Square Orbits: The Flaw Reexamined," *Historia Mathematica*, **19**, 60-70 (1992) (a copy of which is available online on the ScienceDirect website). As we saw in Theorem 5.9 above (which is Proposition 11 of Newton's Book One in *Principia*, if a body orbits the sun an an ellipse (that is, Kepler's First Law holds), then it is under an inverse square law of attraction. Newton considers a body moving along a hyperbola in his Proposition 12, and a body moving along a parabola in his Proposition 13 (both of Book One). In these other cases, the body is also under an inverse square law of attraction. So Newton's Propositions 11, 12, and 13 cover all possible conic sections. The converse is thought to be addressed in his Corollary 1 to Proposition 13 (Weinstock uses the University of California Press 1962 version of *Principia* translated by Motte and revised by Cajori):

> **Corollary 1.** From the three last Propositions [11, 12, and 13] it follows, that if any body P goes from the place P with any velocity in the direction of any right line PR, and at the same time is urged by the action of a centripetal force that is inversely proportional to the square of the distance of the places from the centre, the body will move in one of the conic sections, having the focus in the centre of force; and conversely." (See page 61.)

Johann Bernoulli (August 6/July 27, 1667–January 1, 1748) publicly called attention to problems with Newton's argument in 1710, and presented a proof of his own using differential calculus (see page 147 of Ostermann and Wanner). Weinstock surveys the objections to Newton's presentation, and addresses objections to his own previous work. He comments about his argument that: "…the thoughtful reader should readily conclude that there exists no basis for the claim that Propositions XI–XIII sum Corollary 1 in *Principia* Book One serve as (even an outline of) a proof that an inverse-square force implies a conic-section orbit." (See his page 65) Of course it is not the *validity* of the claim that is in question, but instead the *validity of Newton's argument* for it; as we stated above, a proof can be given with the material of Calculus 3 (MATH 2110). So this leaves us without a geometric argument (in contrast to an argument based on calculus) for the claim that an inverse square law of attraction results in orbits that are in the shapes of conic sections. We address this in Section 5.10 Supplement. Feynman's Lost Lecture.

Note. We conclude this section with some comments related to Newton's geometric approach to the problem of planetary motion. N. Guicciardini in "Geometry, the Calculus and the Use of Limits in Newton's Principia" (appearing in P. Cerrai, P. Freguglia, C. Pellegrini (eds) *The Application of Mathematics to the Sciences of Nature*, Boston: Springer, 2002) commented in the abstract of his work that: "Nowadays, a student of 'Newtonian mechanics' will find the language used in the post-Eulerian era somewhat familiar. On the contrary, the language of the *Principia*, burdened by geometrical diagrams, the theory of proportions, almost devoid of symbolical expressions, leaves our student, even a tenacious one, perplexed." Newton writes the *Principia* between 1684 (when he wrote has manuscript "On the Motion of Bodies in an Orbit") and 1687 (when *Principia* was published). In the 1660s and 1670s, Newton wrote several manuscripts (which he did not publish at the time) on calculus. However, the first published papers on calculus were two papers published by Gottfried Wilhelm Leibniz (July 1/June 21, 1646–November 14, 1716) in 1684 and 1686; this timing led to the argument between Newton and Leibniz which is described in Bardi's *Calculus Wars.* Newton's first calculus publication appears as an appendix in his 1704 *Opticks.* So Newton uses classical geometric arguments instead of the new (and not well-known) calculus. It is in Leonhard Euler's (April 14, 1707–September 18, 1783) *Mechanics* (in Latin, *Mechanica sive motus scientia analytice exposita*) of 1736 that the power of calculus is applied to mechanical/dynamic problems and it is from this work that we have "Newtonian mechanics" as studied in a calculus-based physics class today (thus Guicciardini's comment about the "post-Eulerian era").

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