Section 5.2. Regiomontanus and Euler's Trigonometric Functions.

Note. In this section we consider the history of the trigonometric functions (concentrating on sine, cosine, and tangent). We define the trig functions using right triangles and the unit circle, and give geometric proofs of the summation formulas (or "addition formulas"). Finally, we discuss some of the exact values of the trig functions. We pick up where we left off in the last section, with the chord function of Ptolemy.

Note. Trigonometry did not advance past the stage of the chord function during the Greek period. As addressed in Section 5.1. Ptolemy and the Chord Function, Ptolemy introduced the chord function, $\text{chord}(\alpha) = 120\sin(\alpha/2)$, and gave a table of value of it for angles between $1/2^\circ$ and 180° and $1/2^\circ$ increments. Though not a trigonometric function itself, it is closely related to the sine function. According to the MacTutor website on The Trigonometric Functions (accessed (6/14/2023):

> "The first actual appearance of the sine of an angle appears in the work of the Hindus. Aryabhata [476 CE–550 CE], in about 500, gave tables of half chords which now really are sine tables and used jya for our sin. This same table was reproduced in the work of Brahmagupta [598–670] (in 628) and detailed method for constructing a table of sines for any angle were give by Bhaskara [1114–1185] in 1150."

This same website reports that the Arabs worked with sines and cosines and Mohammad Abu'l-Wafa (June 10, 940–July 15, 998) by the year 980 knew the trig identity $\sin 2x = 2\sin x \cos x$.

Note. Johann Müller Regiomontanus (June 6, 1436–July 6, 1476) was born in what today is Bavaria, Germany. He studies at the University of Leipzig between 1447 and 1450, and at the University of Vienna between 1450 and 1452. In 1457 the University of Vienna awarded him a master's degree, and he was appointed to the faculty of the University of Vienna.

He studied mathematics and astronomy. In his study of astronomy, he came to appreciate the need for a systematic account of trigonometry. He gave such a work in *De triangulis omnimodis* ("The Properties of Triangles") in 1464, though it was not published in print until 1533. This history and the image above are from MacTutor biographical webpage on Regiomontanus (accessed 6/15/2023). Figure 5.5 gives an image from Regiomontanus' work where he introduces "the unit circle" which we still use today (well, it doesn't look like he assumes radius 1 for his circle) and uses it to define the trigonometric functions.

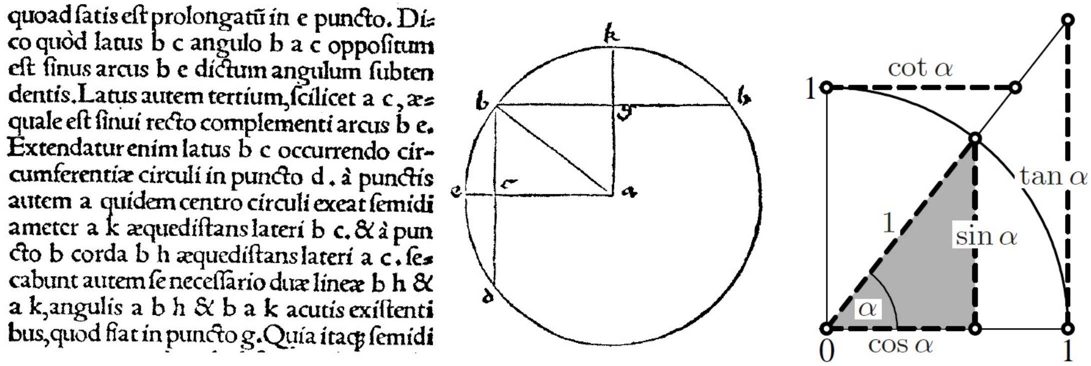

Fig. 5.5. The "trigonometric circle" and definitions of the trigonometric functions. Left: First publication by Regiomontanus, 1464, printed in 1533

Note. For a survey of the history of trigonometry, we turn to Ivor Grattan-Guinness' "An Overview of Trigonometry and Its Functions," in I. Grattan-Guinness (ed.), *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Volume 1 (London, 1994), 499–503. In the first paragraph of this work, it is stated that "the history of trigonometry is not well represented in the literature…" This situation has changed some recently. In the 300 page *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry* by Glen Van Brummelen (Princeton University Press, 2009) a detailed history of trigonometry is given for the Egyptians and Babylonians, Greeks, Indians, Arabs, and Europeans up to, and including, Regiomontanus. The same author also wrote the 300 page *The Doctrine of Triangles: A History of Modern Trigonometry* (Princeton University Press, 2021) which gives a detailed history of trigonometry from the time of Regiomontanus to the time of Legendre and Fourier years (in the early 1800s), including some history of Chinese trigonometry. The main motivation for the study of trigonometry was astronomy. We have mentioned Ptolemy's contributions in the *Almagest*. Both Indian and Arabic mathematicians continued the study of mathematical properties of trigonometric quantities, though their results were unknown in medieval Europe. In addition to the astronomical applications, there are the obvious use of trigonometric quantities in the determination of the height of an object based on the length of its shadow, and similar applications related to sundials. Related to these ideas are calculations of heights of buildings and mountains, and surveying. The role of trigonometry grew in Europe in the Late Middle Ages (1300–1500) to the point that is was viewed as a "subject" itself and books devoted to it started to appear. Nicolaus Copernicus' (February 19, 1473–May 24, 1543) *De revolutionibus orbium coelestium* ("On the Revolutions of the Celestial Spheres," 1543), which introduced the idea of a sun-centered universe (reintroduced, technically, since the idea had appeared in the work of Aristarchus of Samos [circa 310 BCE–230 BCE]), included chapters on planar and spherical trigonometry which was first published in 1542 by Copernicus' editor Georg Joachim Rheticus (February 16, 1514–December 4, 1574). This is shortly after the appearance of Regiomentanus' published version of *De triangulis omnimodis*, which also covered both planar and spherical trigonometry and considered both the sine function and its inverse. The term "trigonometry" was first used in the book *Trigonometria* in 1595 by Polish theologian Bartholomeo Pitiscus (August 24, 1561–July 2, 1613).

Image from the MacTutor webpage on Bartholomeo Pitiscus

Trigonometria includes sum and difference formulas for sine and cosine, along with double and triple angle formulas. Although infant algebra was on the scene at this stage, Pitiscus gave his formula in a less-modern style. François Viète (1540–December 13, 1603) introduced the first systematic algebraic notation in his 1591 *In artem analyticam isagoge* ("Introduction to the Art of Analysis," also known as *Algebra Nova* or "New Algebra").

Image from the MacTutor webpage on Viète

Viète began publishing his several-volume work *Canon Mathematicus* in 1571. I was a mathematical introduction to the astronomy, but also covered trigonometry, contained trigonometric tables, and gave details on how to solve both plane and spherical triangles. In 1593 he published a book on trigonometry and geometry, *Supplementum geometriae*. In it he also gave geometrical solutions to doubling a cube and trisecting an angle. Included in his work was a version of a formula for $\sin n\theta$ as functions of $\sin\theta$ and $\cos\theta$. As algebra developed into function theory (in the 17th and 18th centuries), formulas for both planar and spherical trigonometric functions proliferated. In particular, Leonhard Euler (April 15, 1707—September 18, 1783) gave a number of such formulas in his 1748 *Introductio ad analysin infinitorum* ("Introduction to the Analysis of the Infinite"), including in Chapter 8 "Euler's formula" $e^{i\theta} = \cos\theta + i \sin\theta$; thus the "Euler's Trigonometric Functions" part of the title of this section.

Image from the MacTutor webpage on Euler

The webpages in this note were accessed 6/15/2023.

Definition. Consider a right-angled triangle placed in a circle of radius one, $x^2 + y^2 = 1$, in the Cartesian plane, as shown in Figure 5.5 above. We take angle α as having the positive x-axis as its *initial side* and the hypotenuse of the right triangle is its *terminal side* (that is, α is in *standard position*). Let (x, y) be the point on the terminal side of α and on the unit circle. Then define $\cos\alpha = x$ and $\sin\alpha = y$. (Notice that Ostermann and Wanner define these trig functions using *lengths* of sides of the right triangle, but in this way they only are valid for acute angles.)

Define

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} \text{ and } \cot\alpha = \frac{\cos\alpha}{\sin\alpha},$$

for all α for which we do not get division by 0.

Note. Since $x^2+y^2 = 1$, $x = c\cos\alpha$, and $y = \sin\alpha$ then $\cos^2\alpha+\sin^2\alpha = 1$. We saw in the previous section that $\sin\alpha = \frac{1}{2}\text{cord}(2\alpha)$. Of course, a triangle can be scaled so that the hypotenuse has any (positive) length, and then the trig functions can be defined in the usual way in terms of "opposite," "adjacent," and "hypotenuse." We can also geometrically prove the summation formulas, as given next.

Theorem 5.2. (Addition Formulas). The following identities hold:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\cos\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.
\end{aligned}$$

Note. Based on our definition of sine and cosine, we can see that $\cos(-\theta) = \cos\theta$, $\sin(-\theta) = -\sin\theta$, and $\tan(-\theta) = -\tan\theta$, so that the summation formulas also imply the difference formulas:

$$\begin{aligned}
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\cos\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}.
\end{aligned}$$

With $\alpha = \beta$ we get the double angle formulas:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \text{ and } \cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - \sin^2\alpha = 2\cos^2\alpha - 1,$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

With α replace by $\alpha/2$ in the double angle formulas, we get the half angle formulas:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}, \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}, \text{ and } \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

(Each of these is derived from the double angle formulas for cosine.) Geometric proofs of the double and half angle formulas for sine and cosine can be given using Figure 5.8.

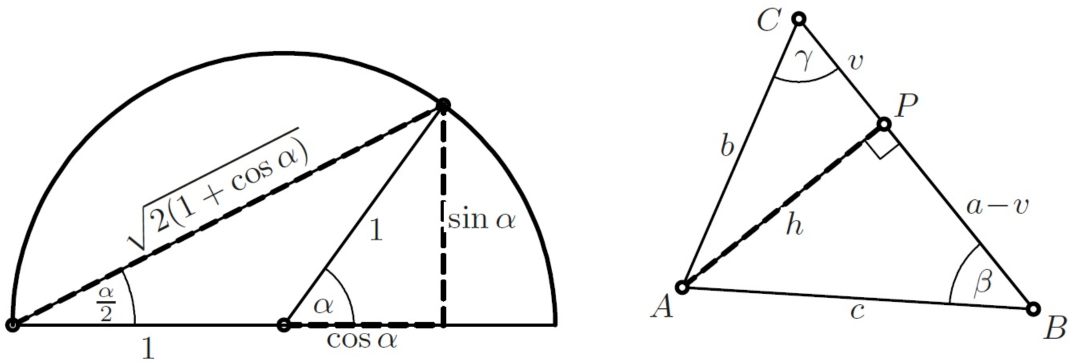

Fig. 5.8. Half angles (left); the law of cosines and the law of sines (right)

Note. We can use the Pythagorean Theorem, an equilateral right triangle, and an isosceles right triangle to find the trig functions of 30°, 45°, and 60° (the "special angles"). Then using the half angle formulas, summation formulas, and difference formulas we can find the trig functions of (for example) 3°, 6°, 9°, 12°, ... Surprisingly, this technique cannot be used to find the exact values of 20°. For more details on this claim, see my online notes for Introduction to Modern Algebra 2 (MATH 4137/5137) on Section VI.32. Geometric Constructions and my associated YouTube video Compass and Straight Edge Constructions (notice Theorem 32.11 and the lemma used to prove it).