Section 5.9. Trigonometric Formulation of Conics

Note. In this brief section, we give a polar coordinate equation for an ellipse and find the area swept out be a line joining a focus on the ellipse to a point on the ellipse.

Note. Let point F be a focus on an ellipse and let P be a point on the ellipse. We denote the angle that $\overline{PF}$ makes with the major axis as φ (see Figure 5.26). Angle φ is the *true anomaly*. Let P_c be the (vertical) projection of point P onto a circle of radius equal to the semimajor axis of the ellipse, as shown in Figure 5.26. The central angle u in the circle determined by the semimajor axis of the ellipse and $\overline{OP_c}$ is the *eccentric anomaly*. Let r be the distance PF.

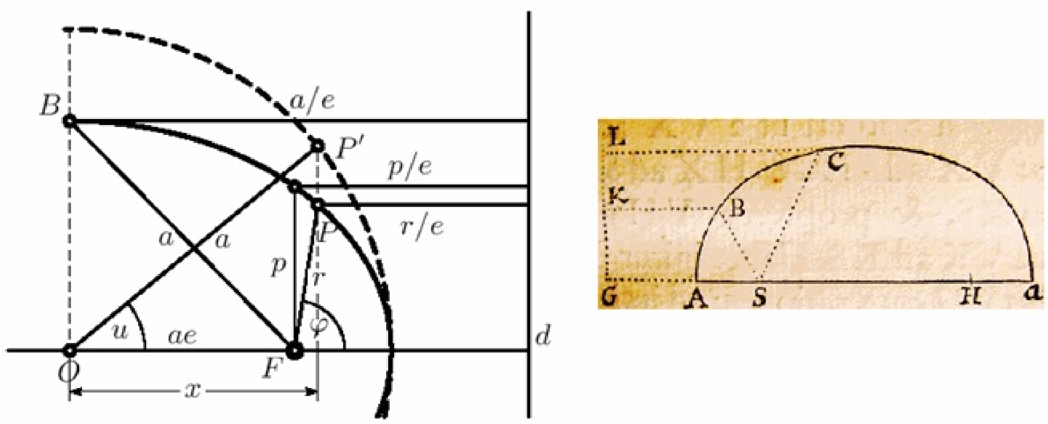

Fig. 5.26. A point P on an ellipse, F the focus (Sun), φ the true anomaly, u the eccentric anomaly, a the semi-major axis, and e the eccentricity (left); illustration from Newton's *Principia* (right)

Theorem 5.9.A. With the parameters introduced above and in Figure 5.26 we have the relations $r = \dfrac{p}{1 + e\cos\varphi}$ and $r = a - ex = a - ae\cos u$ where p is the vertical distance from a focus to the ellipse and x is the directed distance of P from the minor axis of the ellipse (when the ellipse has its major axis horizontal; see Figure 3.4 in Section 3.2. The Ellipse).

Theorem 5.9.B. The area $\mathcal{A}$ swept out by the line joining the focus F to a point P on the ellipse over an angle φ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2}(u - e \sin u).$$

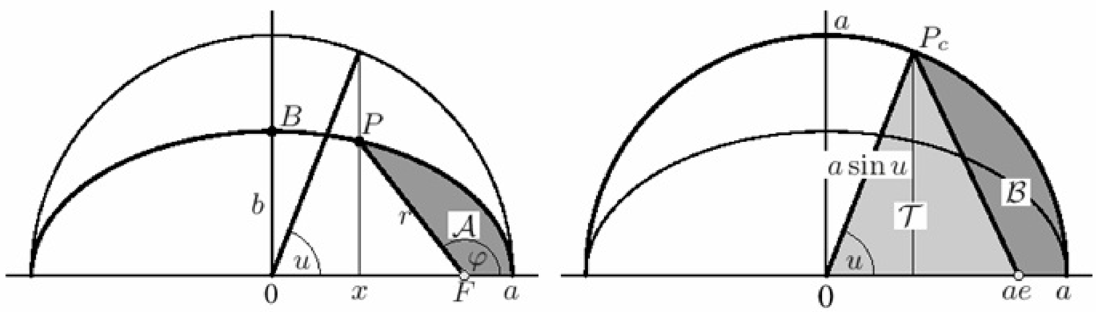

Fig. 5.27. Computation of the area $\mathcal{A}$ swept out by the radius vector