Chapter 3. Conic Sections

Note. In this chapter, we consider parabolas, ellipses, and hyperbolas. We explain how the "Three Famous Problems of Greek Geometry" motivate the study of conic sections. The main character in this history is Apollonius of Perga (circa 262 BCE–circa 190 BCE). He wrote *Treatise on Conic Sections* which, together with Euclid's *Elements*, forms the "cream of the classical period's contributions" (according the a quote of Morris Kline on page 61 of our text book; this quote is from Kline's *Mathematical Thought from Ancient to Modern Times*, Oxford University Press (1972)). The authoritative edition in English is Thomas Heath's *Apollonius of Perga, Treatise on Conic Sections*, Cambridge (1896) (accessed 9/19/2021).

Apollonius of Perga, from MacTutor History of Mathematics Archive biography of Apollonius (accessed 5/11/2023)

Note. Preceding Apollonius was Menaechmus (circa 380 BCE–circa 320 BCE). As stated in Heath's *Apollonius of Perga, Treatise on Conic Sections*: "Thus the evidence so far shows (1) Menaechmus (a pupil of Eudoxus and a contemporary of Plato) was the discoverer of the conic sections, and (2) that he used them as a means of solving the problem of the doubling of the cube." (See his page xix.) According to the MacTutor History of Mathematics Archive biography of Menaechmus: "Menaechmus is famed for his discovery of the conic sections and he was the first to show that ellipses, parabolas, and hyperbolas are obtained by cutting a cone in a plane not parallel to the base."

Note 3.A. Menaechmus' solution to doubling the cube is described by Eutocius in his commentary on Archimedes' *On the Sphere and Circle*. Given numbers a and b, he wants to find the mean proportionals x and y between them; that is, he wants $\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$. With $\frac{a}{x} = \frac{x}{y}$ we get $x^2 = ay$ and with $\frac{a}{x} = \frac{y}{b}$ we have $xy = ab$. We can then express the values of x and y by finding the intersection of the parabola $x^2 = ay$ and the hyperbola $xy = ab$. In particular, with $a = 1$ and $b = 2$ we have that $x^2 = y$ and $xy = 2$, so that $y = 2/x$ and hence $x^2 = y = 2/x$ or $x^3 = 2$ or $x = \sqrt[3]{2}$. As discussed in Section 1.8. Three Famous Problems of Greek Geometry, this construction using conic sections gives us the ability to double the cube. This history motivates Heath to comment (see his page xvii): "There is perhaps no question that occupies, comparatively, a larger space in the history of Greek geometry than the problem of the Doubling of the Cube." The fascination with the doubling of the cube lead to the creation of the new mathematical structures of the conic sections.

Note. As a final observation about Apollonius's work, Heath also observes (see his page vii) that "the influence of Apollonius upon modern text-books on conic sections is, so far as form and method are concerned, practically *nil*." This is not surprising since today conic sections are taught using Cartesian coordinates and algebraic equations, and these were not developed until the 1600s (as will be discussed in Chapter 7 on the history of Cartesian coordinates).

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