Chapter V. Mappings of the Euclidean Plane

Note. In this chapter we consider one to one and onto mappings (i.e., bijections) of the Euclidean plane onto itself. In doing so, we often treat the plane as the complex plane (of the *Gauss plane*) and take advantage of the algebra of complex numbers. In our exploration, we'll apply group theory from algebra since the mapping form a group.

Section 41. Mappings

Note. In this section we define several fundamental mappings of the Euclidean plane onto itself. Throughout, we treat the plane as $\mathbb{C}$.

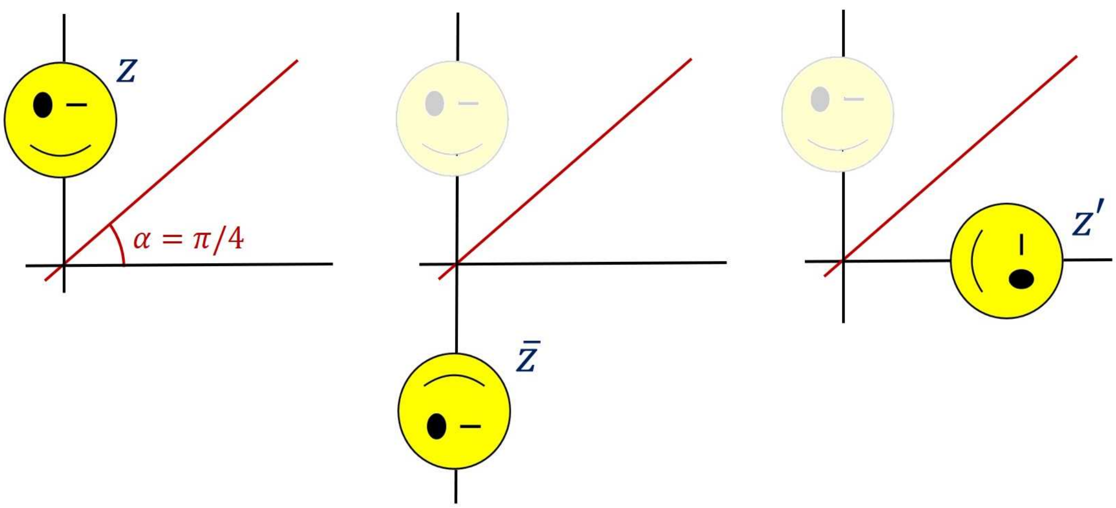
Definition 41.1. Let $b \in \mathbb{C}$. The mapping $z' = z + b$ is a *translation.*

Definition 41.2. Let $\varphi \in \mathbb{R}$. The mapping $z' = (\cos\varphi = i\sin\varphi)z$ is a *rotation about the origin through angle φ.*

Note. Geometrically, a translation by an amount b simply rigidly shifts all of $\mathbb{C}$ by mapping the origin 0 to point b. A rotation fixes the origin 0 and and, as the name suggests, rotates each point about the origin by a change in argument of φ. Notice that a rotation does not change the modulus (that is, $|z'| = |z|$). In fact, a rotation can be represented by multiplication by a complex number of modulus 1; with $a = \cos\varphi + i\sin\varphi$ we have $z' = az$ and $|a| = 1$.

Definition 41.3. A *reflection about the real axis* (or "about the X_1-axis") is the mapping $z' = \overline{z}$. A *reflection about a line* through the origin which makes an angle α with the real axis is $z' = (\cos(2\alpha) + i\sin(2\alpha))\overline{z}$.

Note. The following figure illustrates a reflection about a line:



Notice that a reflection about a line $z' = (\cos(2\alpha)+i\sin(2\alpha))\overline{z}$ is a reflection about the real axis $z' = \overline{z}$ followed by a rotation about the origin through by an angle 2α, $z' = az$ where $z = \cos(2\alpha)+i\sin(2\alpha)$. These mappings must be applied in this order to get the reflection about a line, since if we rotate first we get

$$z' = \overline{(az)} = \overline{(\cos(2\alpha) + i\sin(2\alpha))z} = (\cos(2\alpha)-i\sin(2\alpha))\overline{z} \neq (\cos(2\alpha)+i\sin(2\alpha))\overline{z},$$

unless $-\sin(2\alpha) = \sin(2\alpha)$, that is unless $\sin(2\alpha) = 0$ or $\alpha = n\pi$ for some $n \in \mathbb{Z}$ (in which case the line is in fact the real axis so that the reflection is simply given by conjugation).

Definition 41.4. An *inversion* about the unit circle is the mapping $z' = 1/\overline{z}$.

Note. An inversion is not a mapping of $\mathbb{C}$ onto $\mathbb{C}$ but instead a mapping of $\mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$. In Chapter VI, "Mapping of the Inversive Plane," we resolve this by considering the extended complex plane, $\mathbb{C}^\infty$. For $z = r(\cos\theta_i \sin\theta)$,

$$z' = \frac{1}{\overline{z}} = \frac{1}{r(\cos\theta - i\sin\theta)} = \frac{\cos\theta + i\sin\theta}{r(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)}$$
$$= \frac{\cos\theta + i\sin\theta}{r(\cos^2\theta + \sin^2\theta)} = \frac{1}{r}(\cos\theta + i\sin\theta),$$

so $z' = 1/\overline{z}$ has the same argument as z and its modulus is the reciprocal of the modulus of z.

Definition 41.5. A *dilative rotation* (or "spiral similarity") is a mapping $z' = az$ where $a = |a|(\cos\alpha + i\sin\alpha)$. If z is real then $z' = az$ is a *central dilation* (or "central similarity" or "magnification").

Note. If $z = r(\cos\theta + i\sin\theta)$ then a dilative rotation produces

$$z' = |z|(\cos\alpha + i\sin\alpha)r(\cos\theta + i\sin\theta) = |a|r(\cos(\theta + \alpha) + i\sin(\theta + \alpha)).$$

So z' can be geometrically viewed as first rotating z about the origin through an angle α and then modifying the modulus by a factor of $|a|$ (notice that we could first magnify the modulus and then rotate to get the same z'; that is, these two mapping commute). We follow the books terminology but it would be reasonable to refer to a "dilation" when $|a| > 1$ and a "contraction" when $|a| < 1$.