Section 42. Isometries

Note. We now consider mappings which preserve distances.

Definition. An *isometry* (or "congruent transformation" or "rigid motion") of the Gauss plane (i.e., $\mathbb{C}$) is a one to one mapping of the plane onto itself such that if P is mapped to P' and Q is mapped to Q', the distance PQ equals the distance $P'Q'$, whatever the points P and Q.

Note. In fact, translations, rotations, and reflections are isometries. For a translation by $b \in \mathbb{C}$, we have the distances $|z' - w'| = |(z+b) - (w+b)| = |z - w|$. For rotations about the origin we have the distances

$$|z' - w'| = |(\cos\varphi + i\sin\varphi)z - (\cos\varphi + i\sin\varphi)w| = |(\cos\varphi + i\sin\varphi)(z - w)|$$

$$= |\cos\varphi + i\sin\varphi||z - w| = |z - w|.$$

For reflection about a line through the origin, we have the distances

$$|z' - w'| = |(\cos(2\alpha) + i\sin(2\alpha))\overline{z} - (\cos(2\alpha) + i\sin(2\alpha))\overline{w}|$$

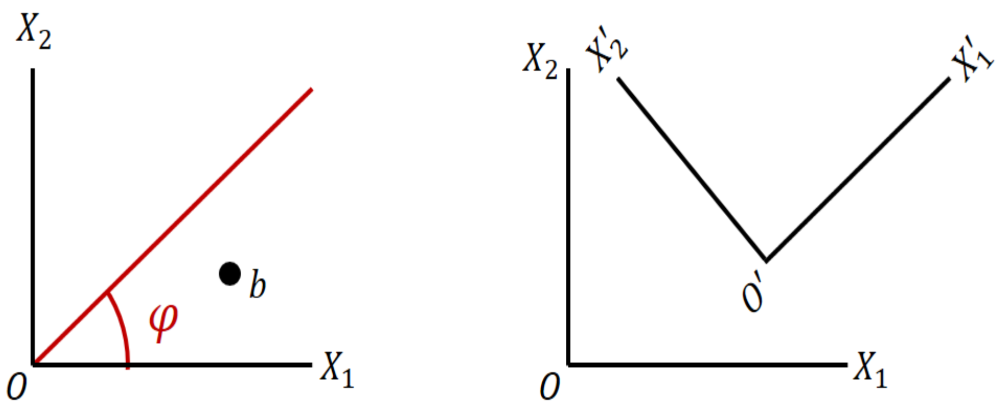
$$= |(\cos(2\alpha) + i\sin(2\alpha))(\overline{z} - \overline{w})| = |\cos(2\alpha) + i\sin(2\alpha)||\overline{z} - \overline{w}|$$

$$= |\overline{z} - \overline{w}| = \left|\overline{(\overline{z} - \overline{w})}\right| = |z - w|$$

since $|z| = |\overline{z}|$.

Note. We now introduce axes to the plane. We take the X_1-axis to coincide with the real axis and the X_2-axis to coincide with the imaginary axis. We can perform a rotation and translation of these axes to get new axes, X_1' and X_2', respectively. A given point in the plane will then have coordinates with respect to both sets of axes.

Note. First we rotate the axes through an angle φ about the origin. We then translate the rotated axes by complex number b.



Consider point z in the X_1X_2-plane (that is, we represent the point in the X_1, X_2 coordinate system as z). The representation of z with respect to the rotated axes is then given by $z(\cos(-\varphi) + i\sin(-\varphi))$. So point b (in the X_1X_2-plane) has become $b(\cos(-\varphi) + i\sin(-\varphi))$. The origin of the X_1X_2-plane is transformed to point b under the rotation and translation. So with O' as the origin of the $X_1'X_2'$-plane, point z has the representation Z' in the $X_1'X_2'$ coordinate system as

$$Z' = z(\cos(-\varphi) + i\sin(-\varphi)) - b(\cos(-\varphi) + i\sin(-\varphi)).$$

If we set $a = \cos(-\varphi) + i\sin(-\varphi)$ and $c = -b(\cos(-\varphi) + i\sin(-\varphi))$ then $|a| = 1$ and $Z' = az + c$.

Note 42.A. We now consider a transformation of the form $Z' = az + c$ where $|z| = 1$ (so here we do not concern ortselves with coordinates and new axes). This transformation is an isometry: $|Z' - W'| = |(az+c) = (aw+c)| = |a||z-w| = |z-w|$.

Note 42.B. Another transformation of interest (we give a geometric interpretation in terms of axes below) is $Z' = a\overline{z} + c$ where $|z| = 1$. This is also an isometry:

$$|Z' - W'| = |(a\overline{z} + c)0(a\overline{w} + c)| = |a||\overline{z} - \overline{w}| = |a| \left|\overline{\overline{z} - \overline{w}}\right| = |a||z - w| = |z - w|.$$

Note. The transformation $Z' = az + c$ where $|a| = 1$ "preserves the orientation" of the axes. That is, the angle from the positive X_1-axis to the positive X_2-axis is the same as the angle from the positive X_1'-axis to the positive X_2'-axis. Both angles are $90°$. The transformation $Z' = a\overline{z} + c$ where $|z| = 1$ "reverses the orientation" of the axes. Under this transformation, the angle from the positive X_1'-axis to the positive X_2'-axis if $-90°$. We'll see in the next section that these two transformations are fundamental isometries of the (Gauss) plane.