Section 43. The Theorem of Isometries

Note. In this section we classify isometries mapping the Gauss plane $\mathbb{C}$ onto itself. We'll see that they fall into two categories, one category contains the orientation preserving isometries and the other contains the orientation reversing isometries. The main result is as follows.

Theorem 43.1. The Main Theorem on Isometries of the Gauss Plane. The set $\mathscr{I}$ of all isometries of the Gauss plane $\mathbb{C}$ (onto itself) is composed of two classes $\mathscr{I}_+$ and $\mathscr{I}_-$. The class $\mathscr{I}_+$ consists of all isometries of the form $z' = az + b$ where $|a| = 1$, and the class $\mathscr{I}_-$ of all isometries of the form $z' = c\overline{z} + d$ where $|c| = 1$.

Definition. Isometries of the Gauss plane $\mathbb{C}$ in $\mathscr{I}_+$ (and so of the form $z' = az + b$ where $|a| = 1$) are called *direct isometries*. Isometries of the Gauss plane $\mathbb{C}$ in $\mathscr{I}_-$ (and so of the form $z' = c\overline{z} + d$ where $|c| = 1$) are called *indirect* (or *opposite*) *isometries*.

Note. We saw in the previous section that $z' = az + b$ and $z' = c\overline{z} + d$, where $|a| = |c| = 1$, are isometries (see Notes 42.A and 42.B). Notice that an isometry is necessarily a one to one mapping (images are the same, that is distance 0 apart, if and only if the inputs are the same). These mappings in $\mathscr{I}_+$ and $\mathscr{I}_-$ are onto since each is invertible: For given $z' \in \mathbb{C}$ let $z = a^{-1}(z' - b)$ so then $az + b = a(a^{-1}(z' - b)) + b = z'$ so all elements of $\mathscr{I}_+$ are onto; for given $z' \in \mathbb{C}$ let $z = \overline{c^{-1}(z' - d)}$ so then $c\overline{z} = c\,\overline{\left(\overline{c^{-1}(z' - d)}\right)} + d = z'$ so all elements of $\mathscr{I}_-$ are onto. We need some other results before proving Theorem 43.1.

Theorem 43.2. An Auxiliary Theorem.

Given two pairs of points z_0, z_1 and w_0, w_1 where $|z_0 - z_1| = |w_0 - w_1| \neq 0$, there is just one mapping of type $\mathscr{I}_+$ and just one of type $\mathscr{I}_-$ which maps z_0 onto w_0 and maps z_1 onto w_1.

Note. We'll see below (Theorem 43.3) that every isometry of the Gauss plane maps lines to lines. First, we give a definition.

Definition. A *collineation* of the Gauss plane is a mapping which preserves collinearity of points, so that straight lines are mapped into straight lines.

Note 43.A. The points z on a line in the Gauss plane $\mathbb{C}$ is of the form $\text{Im}((z - a)/b) = 0$ where the line passes through point a and has "direction vector" b; see my online notes on " Lines and Half Planes in the Complex Plane" at: `http://faculty.etsu.edu/gardnerr/5510/notes/I-5.pdf`.

Note. A collineation may not be an isometry. A central dilation $z' = az$ where $z \in \mathbb{R}$ and $|a| \neq 1$ is a collineation but not an isometry.

Note. For any $z, w \in \mathbb{C}$ we have $|z+w| \leq |z|+|w|$, with equality $|z+w| = |z|+|w|$ if and only if $z = tw$ for some $t \in \mathbb{R}$ where $t \geq 0$. This is the Triangle Inequality for $\mathbb{C}$ (see my online notes "" at `http://faculty.etsu.edu/gardnerr/5510/notes/I-3.pdf`, Theorem I.3.A and Corollary I.3.A). This allows us to prove the following.

Lemma 43.A. For distance $u, v, w \in \mathbb{C}$ we have u, v, w collinear, with v between u and w on the line containing the points, if and only if $|v - u| + |w - v| = |w - u|$.

Theorem 43.3. Isometries are Collineations.
Every isometry of the Gauss plane $\mathbb{C}$ is a collineation.

Note. Since two distinct points z_0 and z_1 determine a line in $\mathbb{C}$, if an isometry is given by its mapping of z_0 and z_1, say w_0 and w_1 as in Theorem 43.2, then we see from Theorem 43.3 that it must map line ℓ containing z_0 and z_1 to line ℓ' containing w_0 and w_1. However, this does not determine the isometry; Theorem 43.2 shows that there are at least two such isometries. Once we prove the Main Theorem (Theorem 43.1) we will see that there are exactly two such mappings is due to how they map the half planes in $\mathbb{C}$ determined by lines ℓ and ℓ'. The mapping in $\mathscr{I}_+$ maps the half plane on the "left side" of ℓ to the "left side" of ℓ', whereas the mapping in $\mathscr{I}_-$ maps the half plane on the "left side" of ℓ to the "right side" of ℓ'. For now, we leave this idea of "left/right side" undefined, but it is related to the ordering of the points u, v, w on ℓ and u', v', w' on ℓ'.

Note. The next result helps illustrate the importance of isometries of $\mathbb{C}$ in the study of Eculidean geometry.

Theorem 43.4. Isometries and Parallel Lines.
An isometry of the Gauss plane $\mathbb{C}$ maps parallel lines onto parallel lines.

Note. In the proof of Theorem 43.4, we have also shown that an isometry of $\mathbb{C}$ maps perpendicular lines to perpendicular lines.

Note. A quicker proof of Theorem 43.4 follows from the fact that if lines ℓ and m are parallel then they do not intersect. Since an isometry is one to one, their images ℓ' and m' cannot intersect and hence are parallel.

Note. We now see how isometries relate to congruent triangles. We first need a preliminary result concerning circles.

Lemma 43.B. If three circles with different centers intersect in two points then the centers of the circles must be collinear.

Theorem 43.5. Determination of an Isometry.
An isometry of the Gauss plane $\mathbb{C}$ is uniquely determined by the assignment of the congruent maps of a given triangle. That is, if z_0, z_1, z_2 are noncollinear points with respective images w_0, w_1, w_2 then for any z in the plane, the image of z is uniquely determined from w_0, w_1, w_2.

Note 43.B. In the proof of Theorem 43.5 we also see that an isometry must map noncollinear points to noncollinear points.

Note. A alternative proof of Theorem 43.5 is to be given in Exercise 43.3 using the fact that an isometry maps parallel lines to parallel lines (Theorem 43.4).

Note. We need one last result before we prove The Main Theorem on Isometries of the Gauss Plane.

Theorem 43.6. There are precisely two isometries of the Gauss plane which map two given points z_0 and z_1 into two given points w_0 and w_1 (respectively) where $|z_0 - z_1| = |w_0 - w_1| \neq 0$.

Note. We can now easily prove the Main Theorem (Theorem 43.1).

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