Section 45. Conjugate and Normal Subgroups

Note. We define normal subgroups and briefly consider group isomorphisms and automorphisms.

Theorem 45.1. If H is a subgroup of G and $g \in G$ then the set of elements $K = g^{-1}Hg$ is a subgroup of G.

Note. We use Theorem 45.1 to justify some definitions that will play an important role.

Definition. Let group G contain elements a and g and let H be a subgroup of G. The element $g^{-1}ag$ is called the *conjugate* of a, and $g^{-1}Hg$ is the *subgroup conjugate* to H.

Definition. A subgroup H of a group G which coincides with every one of its conjugates (that is, $H = g^{-1}Hg$ for all $g \in G$) is called a *normal subgroup* of G. This is also called a *self-conjugate* or *invariant* subgroup.

Theorem 45.2. **A Classification of Normal Subgroups.**
H is a normal subgroup of G if and only if $h \in H$ implies $g^{-1}hg \in H$ for all $g \in G$.

Note 45.A. Theorem 45.2 implies that H is a normal subgroup of G if and only if $g^{-1}Hg = H$ for all $g \in G$, or if and only if $Hg = gH$ for all $g \in G$. That is, H is a normal subgroup if and only if its left and right cosets coincide. In an Abelian group, all subgroups are normal.

Theorem 45.3. Subgroups of Index Two.
A subgroup H of index two in G is always normal.

Definition. Let G be a group with binary operation $\circ$ and let G^* be a group with binary operation $\circ^*$ (we do not mean to imply that the binary operation is function composition, even though we use the same symbol here as is used for function composition). If there exists a bijective mapping between the elements of G and G^* which is such that if a and b in G correspond to a^* and b^* in G^*, then $a \circ b$ in G corresponds to $a^* \circ^* b^*$ in G^* for all $a, b \in G$, then G and G^* are *isomorphic*. This is denoted $G \cong G^*$.

Note. If G and G^* are isomorphic, then they are "structurally the same." The bijective mapping maps the identity of G to the identity of G^* (by Exercise 45.3). If the mapping sends a to a^* the it must send a^{-1} to $(a^*)^{-1}$ (also in Exercise 45.3).

Definition. An automorphism of a group is an isomorphism of the group with itself.

Theorem 45.5. The mapping $\alpha : G \to G$ defined as $\alpha : x' = g^{-1}xg$ where $g \in G$ is fixed, and $x \in G$, is an automorphism of G.

Definition. For group G, the automorphism of Theorem 45.5, $\alpha : x' = g^{-1}xg$, is an *inner automorphism* of G. All other types of automorphisms of G are called *outer automorphisms.*

Revised: 3/2/2019