Section 46. Groups of Mappings

Note. We now consider mappings of the Gauss plane $\mathbb{C}$ as elements of a group.

Theorem 46.1. The translation of the Gauss plane $\mathbb{C}$ form an Abelian group $\mathscr{T}$ which is isomorphic to the additive group of complex numbers. The group operation on $\mathscr{T}$ is composition of mappings.

Theorem 46.2. The Group of Dilative Rotations.
The dilative rotations of the Gauss plane $\mathbb{C}$ about the origin (of the form $z' = az$) form an Abelian group $\mathscr{D}$ (where the binary operation is composition) which is isomorphic to the multiplicative group of the non-zero complex numbers.

Note. We now consider two subgroups of the group $\mathscr{D}$ of dilative rotations.

Theorem 46.3. Groups of Central Dilations and Rotations.
The central dilations (of the form $z' = az$ where $a \in \mathbb{R}$ and $a \neq 0$) form a subgroup $\mathscr{D}^*$ of $\mathscr{D}$. The rotations about the origin (of the form $z' = az$ where $|a| = 1$) form a subgroup $\mathscr{R}_0$ of $\mathscr{D}$. Both $\mathscr{D}^*$ and $\mathscr{R}_0$ are Abelian.

Note. We now address the direct isometries (in $\mathscr{I}_+$) and indirect isometries (in $\mathscr{I}_-$) of $\mathbb{C}$. This will give us a geometric interpretation of normal subgroups.

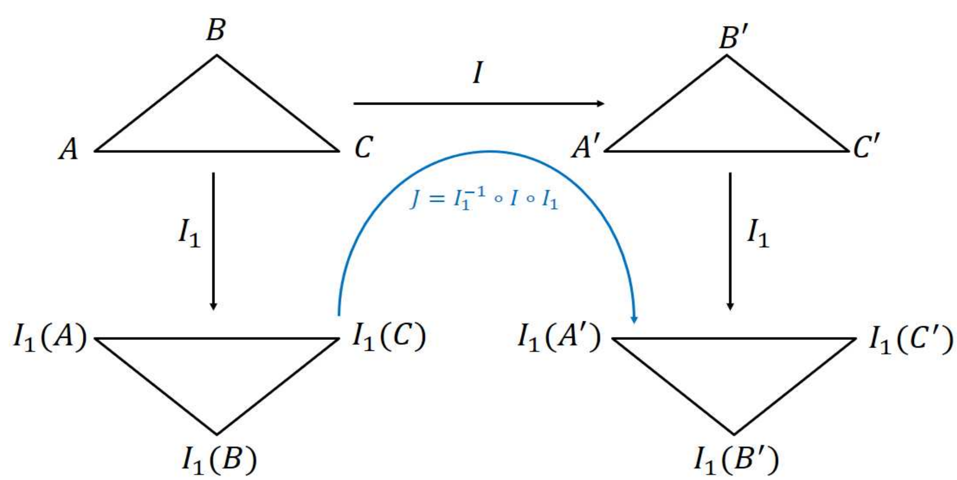
Theorem 46.4. The Group Property of Isometries.

The set $\mathscr{I}$ of isometries of the Gauss plane $\mathbb{C}$ form a group, with the subset $\mathscr{I}_+$ of direct isometries form a normal subgroup. The set $\mathscr{I}_-$ of indirect isometries form a coset with respect to $\mathscr{I}_+$. Both $\mathscr{I}_+$ and $\mathscr{I}$ are non-Abelian groups.

Note. In the following corollary to Theroem 46.4 we use "function notation" as opposed to the notation used by Pedoe.

Corollary 46.4. Let ABC be a triangle and $A'B'C'$ a triangle where $I(A) = A'$, $I(B) = B'$, and $I(C) = C'$ for some direct isometry $I \in \mathscr{I}_+$. If $I_1 \in \mathscr{I}$ is any isometry of the Gauss plane $\mathbb{C}$ then the triangles with vertices $I_1(A), I_1(B), T_1(C)$ and vertices $I_1(A'), I_1(B'), I_1(C')$ are also related by a direct isometry; that is, there is $J \in \mathscr{I}_+$ such that $J(I_1(A)) = I_1(A')$, $J(I_1(B)) = I_1(B')$, and $J(I_1(C)) = I_1(C')$.

Note. The proof of Corollary 46.4 is based on the following diagram:



Pedoe uses this idea to claim: "...the property of being a map of a geometrical figure by an element of a normal subgroup is left unchanged (invariant) under the mappings by elements of this group G." See page 178.

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