Section 47. Similarity Transformations and Results

Note. In Section 0.1, "The Euclidean Plane," triangles are defined)as triples of points) and oriented (or signed) angles are defined. For signed angle $\measuredangle AOB$ we have $\measuredangle BOA = -\measuredangle AOB$. With d as a metric on the Euclidean plane, triangles ABC and $A'B'C'$ are *directly similar* if $d(A', B') = kd(A, B)$, $d(B', C') = kd(B, C)$, $d(C', A') = kd(C, A)$ and $\measuredangle B'C'A' = +\measuredangle BCA$, $\measuredangle C'A'B' = +\measuredangle CAB$, $\measuredangle A'B'c' = +\measuredangle ABC$ for some fixed $k > 0$. Triangle ABC and $A'B'C'$ are *indirectly similar* if $d(A', B') = kd(A, B)$, $d(B', C') = kd(B, C)$, $d(C', A') = kd(C, A)$ and $\measuredangle B'C'A' = -\measuredangle BCA$, $\measuredangle C'A'B' = -\measuredangle CAB$, $\measuredangle A'B'c' = -\measuredangle ABC$ for some fixed $k > 0$. If triangle sABC and $A'B'C'$ are either directly similar or indirectly similar with $k = 1$, then the triangles are *congruent* (notice that we do not distinguish between directly and indirectly congruent). In this section we define similarity transformations (or similitudes) and classify the set of all similitudes of the Gauss plane $\mathbb{C}$ in "The Main Theorem for Similitudes" (Theorem 47.2).

Definition. A one to one and onto mapping of the Gauss plane $\mathbb{C}$ which is such that the distance between any given pair of points is changed in a fixed ratio is called a *similarity transformation* or *similitude*.

Note. Consider the mapping $z' = az + b$ where $a \neq 0$. For $u, v \in \mathbb{C}$ we have

$$|u' - v'| = |(ab + b) - (av + b)| = |a||u - v|$$

so that this is a similitude with $k = |a|$. Consider the mapping $z' = c\overline{z} + d$ where

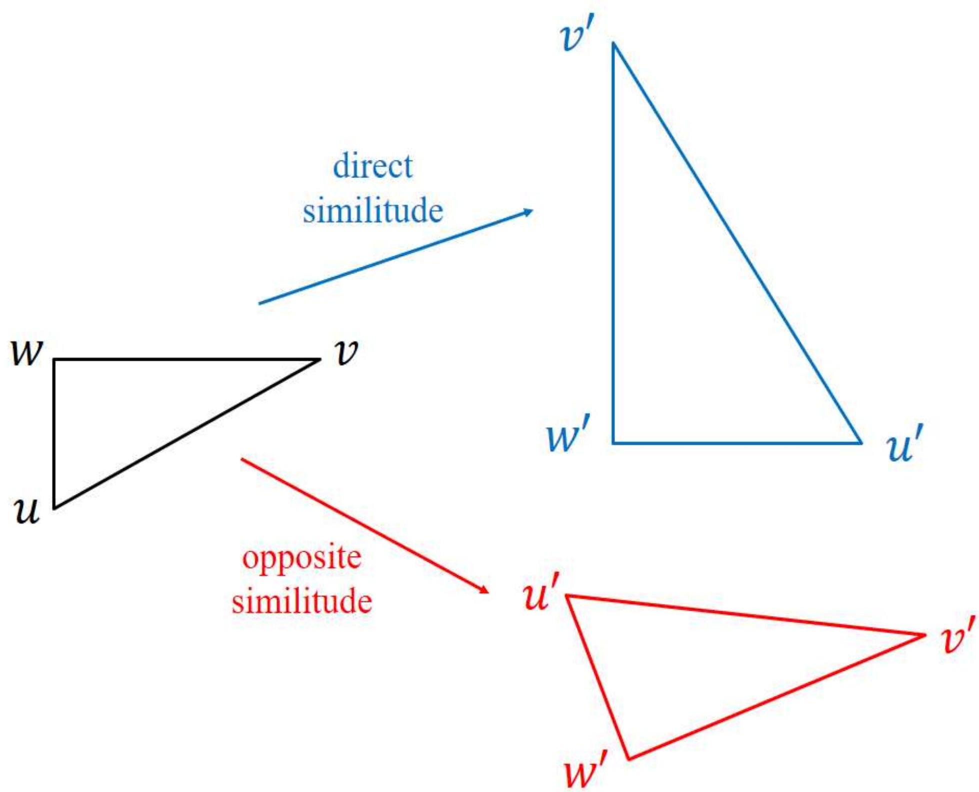
$c \neq 0$. For $u, v \in \mathbb{C}$ we have

$$|u' - v'| = |(c\overline{z} + d) - (c\overline{v} + d)| = |c||\overline{u} - \overline{v}| = |c||u - v|$$

so that this is a similitude with $k = |c|$.

Definition. A similitude of the form $z' = az + b$ where $a \neq 0$ is a *direct similitude*. A similitude of the form $z' = c\overline{z} + d$ where $c \neq 0$ is an *opposite similitude*.

Note. Geometrically, a direct similitude $z' = az + b$ consists of a dilative rotation (the rotation is through an angle $\arg(a)$) and a translation. An opposite similitude $z' = c\overline{z} + d$ consists of a refelction about the real axis (this gives it its "opposite" properties; this reverses oriented angles), a dilative rotation, and a translation.



Note. As with isometries, we can classify similitudes into two categories, direct similitudes and opposite similitudes. This is accomplished in the next theorem which we prove later in this section.

Theorem 47.2. The Main Theorem on Similitudes of the Gauss Plane.
The set $\mathscr{S}$ of all similitudes of the Gauss plane $\mathbb{C}$ is composed of two classes, $\mathscr{S}_+$ and $\mathscr{S}_-$. The class $\mathscr{S}_+$ consists of all similitudes of the form $z' = az + b$ and the class $\mathscr{S}_-$ of all similitudes of the form $z' = c\overline{z} + d$.

Note. The proof of Theorem 47.2 is similar(!) to the proof of The Main Theorem on Isometries of the Gauss Plane, Theorem 43.1. As a consequence, we prove several results similar to those in Section 43. The next result corresponds to Theorem 43.2 and the proof is virtually the same.

Theorem 47.3. An Auxiliary Theorem.
Given two pairs of points z_0, z_1 and w_0, w_1 where $|z_0 - z_1| = k|w_0 - w_1| \neq 0$, there is just one mapping of type $\mathscr{S}_+$ and one of type $\mathscr{S}_-$ which maps z_0 to w_0 and maps z_1 to w_1.

Theorem 47.4. Similitudes are Collineations.
Every similitude of the Gauss plane $\mathbb{C}$ is a collineation.

Theorem 47.5. Similitudes and Parallel Lines.

A similitude of the Gauss plane $\mathbb{C}$ maps parallel lines onto parallel lines.

Note. The proof of Theorem 47.5 is identical to the proof of Theorem 43.4 (for isometries), except that Theorem 43.3 in that proof is replaced with Theorem 47.5 in this proof. We also have (as we did in the proof of Theorem 43.4) that a similitude maps perpendicular lines to perpendicular lines.

Theorem 47.6. Determination of a Similitude.

A similitude of the Gauss plane $\mathbb{C}$ is uniquely determined by the assignment of a map of a triangle which is similar to the given triangle. That is, if z_0, z_1, z_2 are noncollinear points with respective images w_0, w_1, w_2 then for any z in the plane, the image of z is uniquely determined from w_0, w_1, w_2.

Theorem 47.7. There are precisely two similitudes of the Gauss plane $\mathbb{C}$ which map two given points z_0 and z_1 onto the given points w_0 and w_1 where $|w_0 - w_1| = k|z_0 - z_1| \neq 0$.

Note. We now have the equipment of prove The main Theorem on Similitudes of the Gauss Plane (Theorem 47.2). The proof is basically the same as for the corresponding result for isometries (Theorem 43.1).

Note. As with isometries, the set of similitudes form a group, as follows.

Theorem 47.8. Group Properties of Similitudes.

The similitudes form a group $\mathscr{S}$, the direct similitudes forming a normal subgroup $\mathscr{S}_+$. The opposite similitudes form a coset $\mathscr{S}_-$ with respect to $\mathscr{S}_+$. Neither $\mathscr{S}$ nor $\mathscr{S}_+$ is Abelian. The group of isometries $\mathscr{I}$ is a normal subgroup of $\mathscr{S}$, and $\mathscr{I}_+$ is a normal subgroup of $\mathscr{S}_+$.

Note. We see from the proof of Theorem 47.8 that the (left and right) cosets of $\mathscr{I}_+$ in $\mathscr{S}_+$ are of the form $\{az+b \mid a,b \in \mathbb{C}, |a| = K\}$ for some $K > 0$ (in fact, this is the coset $S\mathscr{I}_+$ where $S \in \mathscr{S}_+$ is of the form $cz+d$ with $|c| = K$). The cosets of $\mathscr{I}$ in $\mathscr{S}$ are of the form $\{az+b, a'z+b' \mid a,b,a',b' \in \mathbb{C}, |a| = |a'| = K\}$ for some $K > 0$ (in fact, this is the coset $S\mathscr{I}$ where $S \in \mathscr{S}$ is either of the form $cz+d$ or $c'z+d'$ with $|c| = |c'| = K$). Notice that the cosets are disjoint and their union over all K equals the larger group, as expected.

Note. We can use the algebraic properties of complex numbers to explore the existence of a direct similitude of triangles as follows.

Theorem 47.9. Condition for Direct Similarity of Triangles.

The vertices $z_1, z_2, z_3 \in \mathbb{C}$ of a triangle are mapped by a direct similitude onto the corresponding vertices $w_1, w_2, w_3 \in \mathbb{C}$ of another triangle if and only if

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Note. We can use the fact that the direct similitudes $\mathscr{S}_+$ form a normal subgroup of the group $\mathscr{S}$ of all similitudes to prove the following.

Theorem 47.10. Suppose the points z_1, z_2, z_3 are related to the points w_1, w_2, w_3 by a direct similitude, say $w_i = S_1(z_i)$ for $i = 1, 2, 3$. If S is any direct or indirect similitude, then the triangles with vertices $S(z_1), S(z_2), S(z_3)$ and $S(w_1), S(w_2), S(w_3)$ are also related by a direct similitude.

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