Section 48. Groups of Translations and Rotations

Note. We now consider some subgroups of some of the groups already introduced. We also consider isomorphisms of groups of transformations of the Gauss plane.

Definition. A transformation group of the Gauss plane $\mathbb{C}$ which contains a element which maps the origin 0 onto any given point z_0 is said to be a *transitive group*.

Note. In Exercise 48.1 it is to be shown that a transformation group is transitive if and only if any point w can be mapped to any point w' by some element of the group.

Theorem 48.1. The Group of Translations.
The group $\mathscr{T}$ of translations of the Gauss plane is transitive, and a normal subgroup of the group $\mathscr{I}_+$ of direct isometries.

Definition. If α is a mapping of the Gauss plane $\mathbb{C}$, the point z is said to be an *invariant* (or *fixed*) *point* under α if $\alpha(z) = z$. A line ℓ is said to be an *invariant line* if the points of ℓ are mapped onto points of ℓ under α.

Note. Under the translation $T : z' = z + 5$, then real axis is fixed (as is every line in $\mathbb{C}$ of the form $\{z \mid \mathrm{Im}(z) = k\}$ for some constant k), but no points of $\mathbb{C}$ are fixed under T.

Theorem 48.3. The Fixed Point of a Direct Isometry.
Let I_1 be the direct isometry $z' = az + b$ where $|a| = 1$ and $a \neq 1$. Then I_1 has only one fixed point, given by $w = b(1-a)^{-1}$, and I_1 can be written in the form $z' = a(z-w)+w$, which shows that I_1 is a rotation about w.

Note 48.A. If I_1 is any direct isometry then $I_1 : z' = az + b$ where $|a| = 1$. If I_1 is not a translation then $a \neq 1$. By Theorem 48.3, I_1 has one fixed point and I_1 is a rotation about this fixed point, so we have the following.

Theorem 48.4. Direct Isometries and Rotations.
Every direct isometry which is not a translation is a rotation about the fixed point of the isometry.

Definition. Consider the direct isometry that is not a translation, $I : z' = a(z-w)+w$. With the rotation $R_a : z' = az$ and the translation $T_w : z' = z + w$, so that $T_w^{-1} : z' = z - w$, the *canonical form* of I is $I = T_w \circ R_a \circ T_w^{-1}$. This is also called the *standard form* or *normal form* of I.

Note 48.B. In these notes we primarily use function notation and function composition. Pedoe uses a notation that first considers the point and then applies the transformation from left to right. We then have the relationship between Pedoe's notation and our's as:

$$I(z) = a(z-w)+w = zT_w^{-1}R_aT_w = T_w \circ R_a \circ T_w^{-1}.$$

Definition. A *half-turn* about a point w is a rotation about w through an angle π.

Note 48.C. With $|a| = 1$ and $\arg(a) = \pi$ we have $a = -1$. So a half-turn about w is given by $z' = -(z - w) + w = -z + 2w$.

Theorem 48.5. Rotation Groups.
The set of rotations about the point w form a group $\mathscr{R}_w$, and the groups $\mathscr{R}_w$, for all values of w, are isomorphic.

Corollary 48.5. The rotation groups $\mathscr{R}_w$ form a complete set of conjugate subgroups of $\mathscr{R}_0$ within the group of all isometries $\mathscr{I}$ of the Gauss plane $\mathbb{C}$. That is,

$$\{\mathscr{R}_w \mid w \in \mathbb{C}\} = \{I \circ \mathscr{R}_0 \circ I^{-1} \mid I \in \mathscr{I}\}.$$

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