Section 49. Indirect Isometries and Reflections.

Note. In this section we explore fixed points of indirect isometries and give the canonical form of reflections.

Note 49.A. Consider the indirect isometry $z' = a\overline{z} + b$ where $|a| = 1$. If w is an invariant (fixed) point of the transformation then $w = a\overline{w} + b$ or $\overline{w} = \overline{a}w + \overline{b}$. Hence

$$w = a(\overline{a}w + \overline{b}) + b = a\overline{a}w + a\overline{b} + b = |a|^2 w + a\overline{b} + b = w + a\overline{b} + b.$$

So we must have $a\overline{b} + b = 0$. This necessary condition for the existence of a fixed point is also sufficient, since $w = b/2$ is fixed by such a transformation:

$$a\overline{w} + b = a\overline{b/2} + b = a\overline{b}/2 + b/2 + b/2 = (a\overline{b} + b)/2 + b/2 = b/2 = w.$$

More completely, we have the following.

Theorem 49.2. Indirect Isometries as Reflections.
An indirect isometry $z' = a\overline{z} + b$ has invariant points if and only if $a\overline{b} + b = 0$. If it has an invariant point, it is a line reflection and has a whole line of invariant points. Every line reflection is an indirect isometry.

Definition. Let M be an indirect isometry with a fixed point. A *canonical* (or *normal*) *form* of M is $M = T_w \circ M_a \circ T_w^{-1}$ where $T_w : z' = z + w$ for some fixed point w on the line of reflection of Theorem 49.2, and $M_a : z' = a\overline{z}$ where $|a| = 1$ and $\arg(a)/2$ is the angle the reflection line makes with the real axis.

Note. Pedoe deals with coordinate transformations when discussing canonical forms of indirect isometries with fixed points. He changes from "z coordinates" to "Z coordinates" in which the indirect isometry satisfies $Z' = \overline{Z}$. This is done by mapping the real axis (or the "X-axis") to a new X'-axis which coincides with the reflection line. The imaginary axis (the Y-axis") is similarly mapped to a new Y' axis (or a new imaginary axis) and the conjugation is done with respect to these new real and imaginary axes. With $z' = a\overline{z} + b$, he as $z = Z\sqrt{a} + b/2$ and $z' = Z'\sqrt{z} + b/2$ (see page 191).

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