Section 50. More Isometries and Similitudes.

Note. In this section we further analyze compositions of isometries of the Gauss plane $\mathbb{C}$ and define *glide reflections*. We consider mappings of linear under isometries in Hjelmslev's Theorem and consider fixed points of similitudes.

Definition. A mapping M of the Gauss plane $\mathbb{C}$ onto itself is an *involutory* mapping if $M^2(z) = M \circ M(z) = M(M(z)) = z$ for all $z \in \mathbb{C}$.

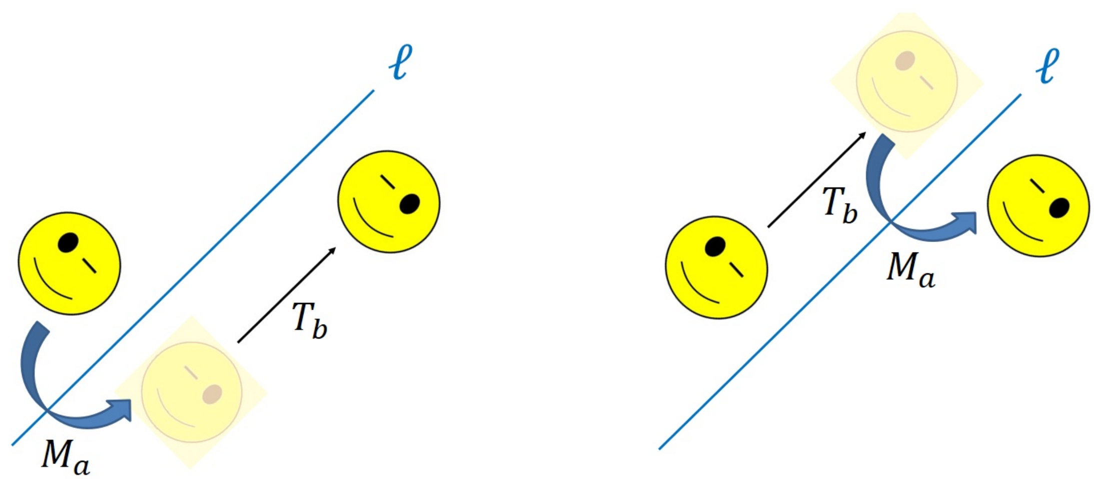
Note. A line reflection is involutory and so is a half-turn about a point (that is, a rotation through an angle π about a point). In fact, these are the only such mappings which are isometries (along with the identity), as we now show.

Theorem 50.1. Every involutory isometry of the Gauss plane $\mathbb{C}$ is either a line reflection, a half-turn, or the identity.

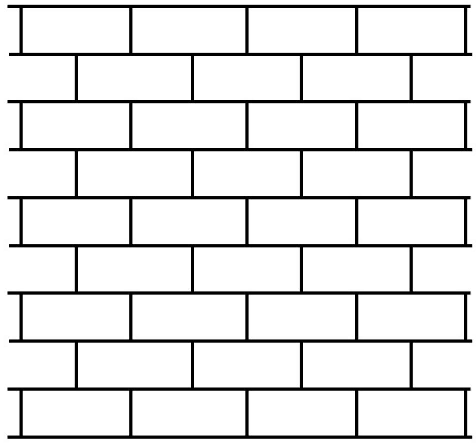
Note. A composition of line reflections is not a line reflection. In fact, we can classify such compositions as follows.

Theorem 50.2. The Composition of Two Reflections.
The composition of reflections in lines ℓ and m result in (i) a translation if and only if the lines are parallel, or (ii) a rotation about the point of intersection if and only if the lines intersect.

Definition/Note 50.A. We now consider opposite isometries without fixed points. If $z' = a\overline{z} + b$ is such a mapping then, by Theorem 49.2, $a\overline{b} + b \neq 0$. With M_a as the line reflection $M_a : z' = a\overline{z}$ and translation $T_b : z' = z + b$, then the opposite isometry is $T_b \circ M_a$. Since this is a reflection followed by a translation, such a mapping will be called a *reflection glide*. If we have commutivity of the reflection and the translation, $T_b \circ M_a = M_a \circ T_n$, then we need $a\overline{(z+b)} = z\overline{z} + b$ or $a\overline{b} = b$ or $a = b/\overline{b}$. Since $\arg(\overline{b}) = -\arg(b)$, then we must have $\arg(a) = 2\arg(b)$ or $\arg(b) = \arg(a)/3$. So the line of reflection makes an angle of $\arg(a)/2$ with the real axis and the translation is in the same "direction" as this line. Geometrically, it should make sense that these two mappings commute:



Definition. In Introduction to Modern Algebra (MATH 4127/5127) you may explore plane isometries in the study of symmetry. The goal is, for a given pattern in the plane, to find all isometries which map the pattern to itself. The collection of such isometries form a group called a *wallpaper group*. Consider this pattern given by brick work:



There are horizontal and vertical lines of symmetries in which reflections about those lines map the pattern onto itself. There are glide reflections related to both horizontal and vertical lines. In addition, there many several half-turn isometries. The group of isometries for this pattern is called "*cmm*." For more details on this, see my online PowerPoint presentation with embedded audio: Groups: A Geometric Introduction. There is also online a transcript and a Windows Media Video version. In addition, the video is on YouTube (last accessed 3/12/2019).

Note. The next theorem shows that any reflection glide is also some glide reflection, even though the original reflection and translation making up the reflection glide may not commute.

Theorem 50.3. A Reflection Glide is also a Glide Reflection.
A reflection in a line ℓ followed by a translation T_b results in an opposite isometry without invariant points if and only if that ℓ is no perpendicular to the position vector b. An opposite isometry without fixed points is equivalent to a glide reflection, that is to a reflection in a line followed by a translation parallel to the line.

Note 50.B. Theorem 50.3 and Theorem 49.2 combine to show that an indirect isometry is either a line reflection (which has a line of invariant points) or a glide reflection (which has no fixed points).

Theorem 50.4.I. Hjelmslev's Theorem.
Suppose the points P on a line are mapped by a plane isometry onto the points P' of another line. Then the midpoints of the line segments PP' either coincide or are distinct and collinear.

Note. Johannes Hjelmslev (April 7, 1873-February 16, 1950) was a Danish mathematician who worked in geometry and the history of geometry. He gave an axiomatic foundation for general plane geometry with no assumption about continuity or parallels.

This information and image are from Harald Borh, Johannes Hjelmslev in Memoriam: 1873–1950, *Acta Mathematica*, **83**, vii-ix (1950).

Note. In the proof of Hjelmslev's Theorem (Theorem 50.4.I) we have proved a more general result concerning the ratio $k : k'$, as follows.

Theorem 50.4.II. The points P on a line are mapped by an isometry onto the points P' of another line. Then the points which divide the segments PP' in the fixed ratio $k : k'$ either coincide, or they are all distinct, collinear, and trace out a range similar to the ranges $\{P\}$ and $\{P'\}$. Coincidence is only possible if the given lines are parallel, and $k = k'$.

Note/Definition. Recall from Theorem 47.2 that the similitudes of the Gauss plane $\mathbb{C}$ fall into two classes, $\mathscr{S}_+$ (of the form $z' = az + b$, called direct similitudes) and $\mathscr{S}_-$ (of the form $z' = c\overline{z} + d$, called indirect or opposite similitudes). We now consider similitudes that are not isometries, called *proper similitudes*. So we consider $z' = az + b$ and $z' = c\overline{z} + d$ where $|a| \neq 1 \neq |c|$.

Theorem 50.5. A proper direct similitude $z' = az+b$, where $|a| \neq 1$, has a unique fixed point. A proper indirect similitude $z' = c\overline{z} + d$, where $|c| \neq 1$, has a unique fixed point.

Note. With w as the fixed point of proper direct similitude $z' = az + b$, we then have $z' - w = (az + b) - (aw + b) = a(z - w)$. So if we "move" the coordinate axes parallel to themselves (that is, without rotation) to pass through the fixed point w then, with the new coordinates of z and z' as Z and Z', respectively (so that $Z = z - w$ and $Z' = z' - w$), then we have $Z' = aZ$. Since $|a| \neq 1$ this is a rotation through an angle $\arg(a)$ and stretching/compression by an amount $|a|$. If we denote the fixed point as O as in Figure 50.5, the point P is stretched to P', where $|OP'| = k|OP|$, $k = |a|$, and then P' is rotated about O through a fixed angle $\arg(a)$.

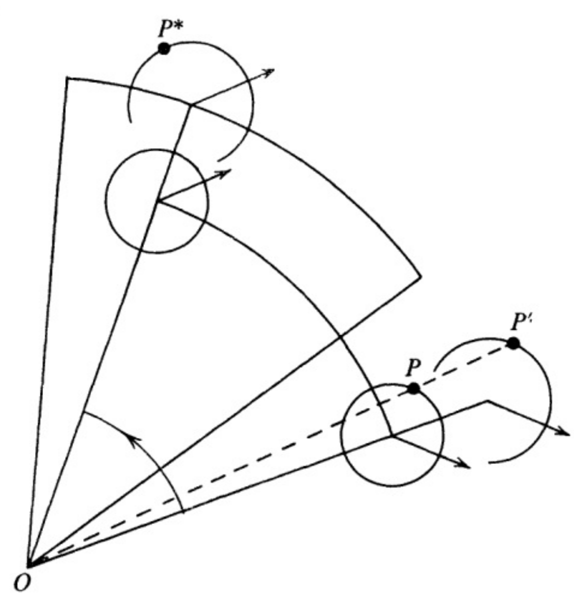


Figure 50.5

Note. With w as the fixed point of proper indirect similitude $z' = c\overline{z} + d$, we then have $z' - w = (c\overline{z} + d) - (c\overline{w} + d) = c(\overline{z - w})$. So if we move the coordinate axes parallel to themselves to pass through the fixed point w then, with the new coordinates of z and z' as Z and Z', respectively (so that $Z = z - w$ and $Z' = z' - w$), then we $Z' = c\overline{Z}$. If we denote the fixed point as O as in Figure 50.6, the point P is reflected in the line through O which makes and angle $\arg(c)/2$ with the X-axis, and the resulting point P' is stretched from O to P'' so that $|OP''| = k|OP'|$, where $k = |c|$.

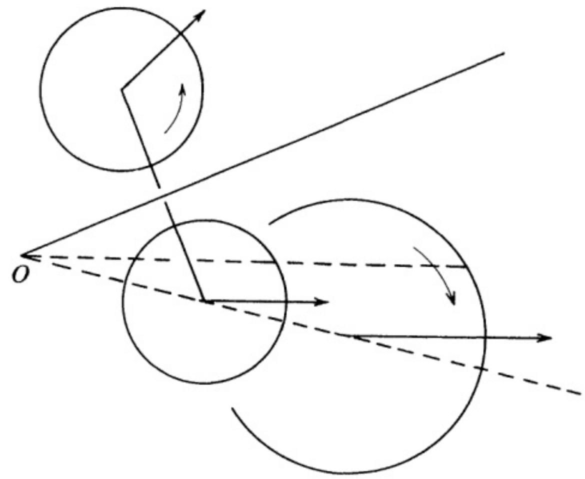


Figure 50.6

Definition. A mapping of the form $z' = az$ is a *dilative rotation* or a *spiral similarity* (as we did in Definition 41.5). A mapping of the form $z' = a\overline{z}$ is a *dilative reflection.*

Note. We have by the previous two results, that every proper direct similarity is a dilative rotation and every indirect similutude is a dilative reflection (though in both cases we need a change of coordinates using the unique fixed point of Theorem 50.5)

Revised: 1/5/2022