Section 51. Line Reflections and Isometries as Reflections

Note. In this section we further consider reflections. We state and prove four theorems. The theorems concern compositions of line reflections with each other (Theorem 51.1) and with reflections about a point (Theorem 51.3), classification of lines as perpendicular in terms of compositions of reflections in the lines (Theorem 51.2), and classify isometries in terms of reflections about lines (Theorem 51.4).

Theorem 51.1. Every product (i.e., composition) of three line reflections is either a line reflection or a glide reflection.

Note. In Theorem 50.2, we saw that the composition of reflections in lines ℓ and m result in (i) a translation if and only if the lines are parallel, or (ii) a rotation about the point of intersection if and only if the lines intersect. In Theorem 14.3, the translation is given in terms of the distance between the parallel lines is given. In Theorem 14.2, the center and angle of the rotation in case (ii) are given. The next theorem gives a necessary and sufficient condition for two lines to be perpendicular in terms of the rotation about the point of intersection. Recall that mapping of the Gauss plane $\mathbb{C}$ onto itself is involutory it its square (i.e., the composition of mapping with itself) is the identity.

Theorem 51.2. Condition for Two Lines to be Perpendicular.
Two distinct lines ℓ and m are perpendicular to each other if and only if the product (i.e., composition) of the reflections in ℓ and m is involutory, and not the identity.

Note. Next, we consider compositions of reflections in a point and a line and classify when they are involutory.

Theorem 51.3. The product (i.e., composition) of the reflections in a point w and in a line ℓ is involutory if and only if w lies on ℓ.

Note. In Exercises 14.1 and 14.2, a geometric proof is to be given that a translation can be written as the product (composition) of reflections in parallel lines, and that a rotation can be written as the product of reflections in two lines through the center of rotation. The next theorem gives algebraic proofs of these claims.

Theorem 51.A. A translation can be written as the product (composition) of reflections in parallel lines, and a rotation can be written as the product of reflections in two lines through the center of rotation.

Note. Finally, we show that every isometry is a composition of reflections.

Theorem 51.4. Isometries as Reflections.
Every isometry is the product (i.e., composition) of at most three reflections. If the isometry has a fixed point, at most two line reflections produce the isometry.

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