Section 52. Möbius Transformations

Note. In this section we define a Möbius transformation from the inversive plane (i.e., the extended complex plane) to itself. We consider the "determinant" of such a transformation, prove that a Möbius transformation maps the set of lines and circles in the Gauss plane to itself (in Theorem 52.3), "dissect" a Möbius transformation into more elementary transformations, and prove that the set of all Möbius transformations form a group under function composition (in Theorem 52.5).

Definition. The transformation mapping $z \mapsto z'$ (where $z \in \mathbb{C}$ and $z \neq -d/c$) given by $z' = \dfrac{az+b}{cz+d}$ where a, b, c, d are complex numbers where $\Delta = ad - bc \neq 0$, is a *Möbius transformation*. The complex number Δ is the *determinant* of the mapping.

Definition. We now extend a *Möbius transformation to the inversive plane*. Define

$$z' = a/c \quad \text{if} \quad z = \infty \tag{1}$$

$$z' = \infty \quad \text{if} \quad z = -d/c \tag{2}$$

In the first case, we replace a/c with ∞ if $a \neq 0$ and $c = 0$. In the second case, we replace $-d/c$ with ∞ when $c = 0$. In both cases, these conditions give $\infty \mapsto \infty$.

Note. You are familiar with the term "determinant" from linear algebra. Recall that a square matrix A is invertible if and only if the determinant of A is nonzero. See my online notes for Linear Algebra (MATH 2010) on Section 4.2. The Determinant of a Square Matrix; see Theorem 4.3, "Determinant Criterion for Invertibility. We will see below that the same property holds for Möbius transformation. In fact, the association

$$\frac{az+b}{cz+d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be used to associate a Möbius transformation with a 2×2 invertible matrix with complex entries. If we represent $z = z_1/z_2 \in \mathbb{C}$ as $[z_1, z_2] \in \mathbb{C}^2$ then we have

$$\frac{az+b}{cz+d} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix}.$$

This is explored more in my online Complex Analysis notes on Section III.3. Analytic Functions as Mapping, Möbius Transformations (see Exercise III.3.26 in John Conway's *Functions of One Complex Variable I*, Second Edition, Springer [1978], on which these notes are based).

Note 52.A. The Möbius transformation $z' = \dfrac{az+b}{cz+d}$ is invertible and has inverse

$$z = \frac{-dz'+b}{cz'-a},$$

as is easily shown.

Note/Definition. In Section 18 of the text book, it is shown that for any two distinct points A and A' in the Euclidean plane, the set of all points P satisfying the equation $|AP| = \lambda|A'P|$ (where λ is a positive constant and $|PQ|$ represents the distance between points P and Q) is a circle when $\lambda \neq 1$ and is a line when $\lambda = 1$. These points make up the *Apollonius circle* (even when $\lambda = 1$ and we have a line). The converse also holds in that for any circle or line there are two points A and A' and a positive λ such the circle or line consists of all points P satisfying $|AP| = \lambda|A'P|$. In the Gauss plane, we can then represent an Apollonius circle as the set of complex numbers z satisfying $|z - p| = k|z - q|$ where k is a positive constant and p and q are complex constants (points p and q are *inverse points* with respect to the Apollonius circle $|z - p| = k|z - q|$). We use this representation of circles and lines to prove the next theorem.

Theorem 52.3. A Möbius transformation is a circular transformation, that is it maps the set of circles and lines into the set of circles and lines.

Note. We see Theorem 52.3 in Complex Analysis 1 in Section III.3. Analytic Functions as Mapping, Möbius Transformations (see Proposition II.3.16). In that class, the term "circle" includes both circles and lines. Sometimes the term "cline" is used to represent the collection of circles and lines; see my online notes on Supplement. Transformations of $\mathbb{C}$ and $\mathbb{C}_\infty$—An Approach to Geometry (a supplement on transformational geometry I use in Complex Analysis 1; see Definition H.3.2.2).

Note. In the proof of Theorem 52.3, we see that the image of Apollonius circle $|z - p| = k|z - q|$ under a Möbius transformation is the Apollonius circle $|z - p'| = k|z - q'|$ where p' and q' are the images of p and q, respectively, under the Möbius transformation. So we also have that inverse points p and q with respect to $|z - p| = k|z - q|$ are mapped to the inverse points p' and q' with respect to $|z - p'| = k|z - q'|$. Hence, we have the following corollary to Theorem 52.3 (we reword Pedoe's statement and refer to a "circle or line" as an "Apollonius circle").

Corollary. The map of an Apollonius circle and a pair of inverse points under a Möbius transformation is an Apollonius circle and a pair of inverse points (where if the circle is a line, the points are mirror images in the line).

Definition. A Möbius transformation of the form $z' = z + a$ is a *translation*. If $z' = az$ where $z > 0$ then the transformation is a *dilation*. If $z' = e^{i\theta}z$ then the transformation is a *rotation*. If $z' = 1/z$ then the transformation is an *inversion*.

Note. The previous definitions are from my online notes for Complex Analysis 1 (MATH 5510) on Section III.3. Analytic Functions as Mapping, Möbius Transformations. Pedoe takes a slightly different definition of "inversion" and, with his definition, $1/z$ is not an inversion, but instead is an inversion followed by a reflection about a line (see Pedoe's page 212).

Note. The special Möbius transformations of the previous definition are fundamental in that every Möbius transformation is a composition of the previously defined types of transformations. We show this in the next theorem; we also address this in the Complex Analysis 1 notes metioned in the previous Note in Proposition III.3.6.

Theorem 52.A. Every Möbius transformation is a composition of translations, dilations, rotations, and inversions.

Note. We now consider the collection of Möbius transformations and show that they form the algebraic structure of a group.

Theorem 52.5. Möbius transformations form a group $\mathscr{B}$ under composition of mappings. If B and C are two Möbius mappings, Δ_B and Δ_C their determinants, then $\Delta_{BC} = \Delta_B \Delta_C$ is the determinant of the Möbius mapping BC.

Note. In a detailed exploration of the group of Möbius transformations, we might consider some of the following. Each is an exercise from John Conway's *Functions of One Complex Variable I*, Second Edition, Springer (1978):

1. Let $GL_2(\mathbb{C})$ be the group of all invertible 2×2 matrices with entries in $mathbbC$ (this is a "general linear" group) and let $\mathscr{B}$ be the group of Möbius transformations. Define $\varphi : GL_2(\mathbb{C}) \to \mathscr{B}$ by $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dfrac{az+b}{cz+d}$. Then φ is a group homomorphism of $GL_2(\mathbb{C})$ onto $\mathscr{B}$. (Exercise III.3.26(a))

2. Let $SL_2(\mathbb{C})$ be the subgroup of $GL_2(\mathbb{C})$ consisting of all matrices of determinant

1 (this is a "special linear" group). Then the image of $SL_2(\mathbb{C})$ under φ (given above) is all of $\mathscr{B}$. (Exercise III.3.26(b))

3. The group $\mathscr{B}$ of all Möbius transformations is a simple group (that is, a group with no proper, nontrivial subgroups). (Exercise III.3.27)

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