Section 53. Cross-Ratio

Note. In this section we define the "cross-ratio" of an ordered 4-tuple of complex numbers. This value is related to Möbius transformations and can be used to determine if the four points are colinear (or "cocircular").

Note. We start with results that lead us to conditions on a Möbius transformation that uniquely determine the transformation.

Theorem 53.1. If a Möbius transformation has more than two distinct fixed points, then it is the identity mapping, $z' = z$.

Note. A consequence of Theorem 53.1 is that a Möbius transformation is uniquely determined by its values on three different points, as shown in the following result.

Theorem 53.2. A Möbius transformation is uniquely determined by the assignment of three distinct points z_j and their three distinct image points z'_j ($j = 1, 2, 3$).

Note. The proof of Theorem 53.2 (in particular, in the Möbius transformations B and C) is based on the idea of a cross-ratio (though this was not mentioned in the proof).

Definition. The *cross-ratio* of the four points (or complex numbers) z_j ($j = 1, 2, 3, 4$) is the complex number

$$(z_1, z_2; z_3, z_4) = \left(\frac{z_1 - z_3}{z_2 - z_3}\right) \Big/ \left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}.$$

Note. Order matters in the cross-ratio, but there is a certain symmetry between the first two entries and the last two entries (as suggested by the use of a semicolon to separate entries two and three).

Theorem 53.3. The cross-ratio satisfies $(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3)$.

Note. A cross-ratio is unaffected by a Möbius transformation, in the following sense.

Theorem 53.4. The cross-ratio of four points is an invariant under Möbius transformations.

Note. One application of the cross-ratio is to recognize collinear/cocircular points, as follows.

Theorem 53.5. The cross-ratio of four points is real if and only if the four points lie on a straight line or a circle.

Note. Pedoe concludes this section by observing that in the inversive plane, we can simply think of "lines" as circles through the point ∞ (whereas traditional circles do not pass though ∞).

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