Section 54. Circles and Conformality

Note. In this section we consider images of circles under Möbius transformations. In particular, we consider Möbius transformations that map the unit circle $|z| = 1$ to itself and maps the interior of the unit circle to itself.

Note. The proof of the next result is somewhat informal and we take some liberties with the idea of the "interior" and "exterior" of a circle. To rigorously deal with this, we need the idea of a simple closed curve. This is addressed in an ambitious version of Introduction to Topology (MATH 4357/5357); see my online notes for algebraic topology (in particular, Sections 61 and 63). None of the claims we make are surprising, but they are not purely rigorous. Exercise 54.3 deals somewhat more clearly with the idea of the "inside" of a circle in the Cartesian plane.

Theorem 54.1. If a circle $\mathscr{C}$ is mapped onto a circle $\mathscr{C}'$ by a Möbius transformation, then the interior of $\mathscr{C}$ is mapped onto the interior of $\mathscr{C}$, or onto the exterior of $\mathscr{C}'$.

Note. We now consider Möbius transformations which map the unit circle $\mathscr{C}$, $|z| = 1$, onto itself and map the interior of the unit circle to itself. Such transformations determine the geometry in the Poincare disk model for hyperbolic geometry, which we explore in Section 56. The Poincare Model of Hyperbolic Geometry. It should be pretty clear that a composition of such Möbius transformations is again such a

Möbius transformation. Therefore, the set of all such Möbius transformations forms a subgroup of the group $\mathscr{B}$ of all Möbius transformations (see Theorem 52.5). The next result gives the form of some of these transformations.

Theorem 54.A. A Möbius transformation of the form $Z = \dfrac{az+b}{\overline{b}z+\overline{a}}$ where $|b| < |a|$ maps the unit circle $|z| = 1$ to itself and maps the interior of $|z| = 1$ to itself.

Note 54.A. In John Conway's *Functions of One Complex Variable I*, Second Edition, Springer (1978), Exercise II.3.10 classifies all transformations mapping $|z| = 1$ to $|Z| = 1$, and mapping the interior of $|z| = 1$ to the interior of $|Z| = 1$. These transformations are all of the form

$$Z = e^{i\theta}\left(\frac{z-\alpha}{\overline{\alpha}z - 1}\right)$$

where $|\alpha| < 1$ and $\theta \in [0, 2\pi)$. Notice that if we replace b with $-b$ in the transformation of Theorem 54.A, and we then divide the numerator and denominator by a, we get

$$Z = \frac{z - b/a}{-(\overline{b}/a)z + \overline{a}/a} = \left(\frac{-1}{\overline{a}/a}\right)\frac{z-b/a}{(\overline{b}/\overline{a})z - 1} = e^{i\theta}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$$

where $\alpha = b/a$ (since $|b| < |a|$ then $|\alpha| < 1$) and θ is the argument of $-a/\overline{a}$. With these transformations classified, we can now prove algebraically that they form a group.

Theorem 54.B. The set of all Möbius transformations transformations mapping $|z| = 1$ to itself and mapping the interior of $|z| = 1$ to itself forms a subgroup of the group $\mathscr{B}$ of all Möbius transformations.

Note/Definition. A function that maps a subset of $\mathbb{C}$ into $\mathbb{C}$ in such a way that it "preserves angles," is called a *conformal map*. The next theorem addresses the conformality of Möbius transformations in the setting of angles between intersecting circles. More generally, we address conformality using smooth paths. An analytic function of a complex variable is conformal at all points of its domain where the derivative is nonzero (see my online notes for Complex Analysis 1 [MATH 5510] on Section III.3. Analytic Functions as Mapping, Möbius Transformations; see Theorem III.3.4).

Theorem 54.3. Suppose circles $\mathscr{C}$ and $\mathscr{D}$ intersect at the point z_0, where u is a unit tangent vector to $\mathscr{C}$ at z_0 and v is a unit tangent vector to $\mathscr{D}$ at z_0. If $\mathscr{C}$ and $\mathscr{D}$ are mapped by a Möbius transformation onto the circles $\mathscr{C}'$ and $\mathscr{D}'$, respectively, which intersect at Z_0 with u mapped to u' and v mapped to v', then u' and v' are tangent to circles at Z_0 to $\mathscr{C}'$ and $\mathscr{D}'$, respectively, and the angle between u' and v' is equal to that between u and v both in measure and in sense. See Figure 54.1.

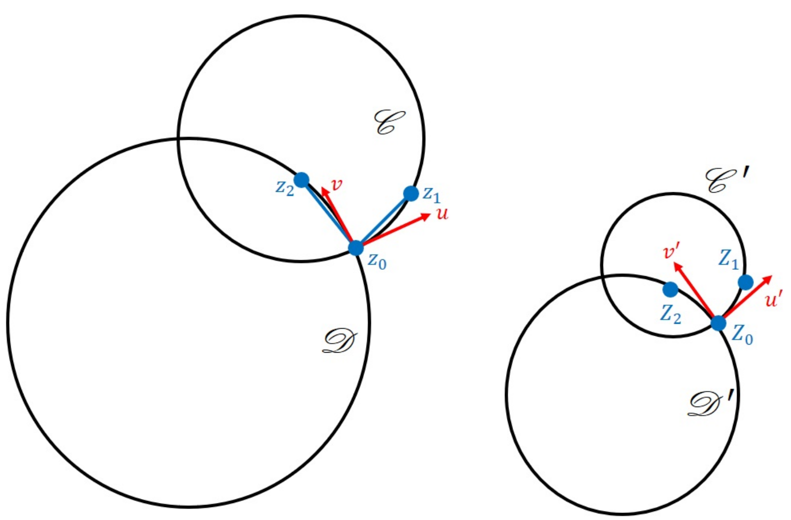

Figure 54.1 (slightly modified)

Note. In the proof of Theorem 54.3 given by Pedoe, complex numbers are treated as vectors in $\mathbb{R}^2$ (which is acceptable; the real part of the complex number and the first component of the vector are the same, as is the imaginary part and the second component). In his proof he derives the a relationship of the form

$$\frac{Z_0 - Z_1}{Z_0 - Z_2} = \frac{(z_0 - z_1)(cz_2 + d)}{(z_0 - z_2)(cz_1 + d)}. \qquad (*)$$

In this, he takes limits as $z_1 \to z_0$ and $z_2 \to z_0$. The right-hand term of $(*)$ then has a limit of 1. Pedoe concludes that "in the limit" $\dfrac{Z_0 - Z_1}{Z_0 - Z_2} = \dfrac{z_0 - z_1}{z_0 - z_2}$. This is true, but that does not mean that there are particular values of z_1 and z_2 for which this equality holds. He then uses this equation in Theorem 47.9 to draw the conclusion that triangles $z_0 z_1 z_2$ and $Z_0 Z_1 Z_2$ are directly similar, from which the conclusion follows. Your humble instructor questions the validity of the proof given by Pedoe. None-the-less, the result is valid and is shown in the Complex Analysis class. A proof of Theorem 54.3, based on the dissection of a Möbius transformation as given in Theorem 52.A, is also to be given in Exercise 54.1.

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