Section 55. M-Transformations

Note. In this section we further explore the set of Möbius transformations, $z \mapsto Z$, which map $|z| = 1$ to $|Z| = 1$ and map the interior of $|z| = 1$ to the interior of $|Z| = 1$. These are the "M-transformations." In particular, we consider such transformation that map a given point and direction to another point and direction (in Theorem 55.2), and consider such transformations that interchange two given points (in Theorem 55.3).

Note. We denote the unit circle $|z| = 1$ as ω, and denote its interior $|z| < 1$ as Ω. The purpose of this chapter is to set up a geometry on Ω. The points will simply be the points in Ω. The lines will be circles that intersect ω at right angles, or diameters of ω (that is, lines that intersect ω at right angles). We'll see that the group of M-transformations corresponds to the group of direct isometries of the Euclidean plane. Recall that direct isometries are of the form $z' = az + b$ where $|a| = 1$ and are in the class denoted $\mathscr{I}_+$. These are the isometries that do not involve a reflection. See Theorem 43.1. The resulting geometry is called "hyperbolic geometry."

Definition. A Möbius transformation that maps ω to ω, and maps Ω to Ω is an *M-transformation.*

Note. We can speak of the "direction" of line or circle at a point as a vector tangent to the line or circle at the point. This is best dealt with by parameterizing the line or circle and then differentiating with respect to the parameter an normalizing the derivative (so that the direction can be expressed as a modulus one complex number, where the real an imaginary parts are interpreted as components of a unit direction vector). Parameterizing a line or circle also gives an orientation to the line or circle.

Theorem 55.2. Let X and A be two points of Ω and let ξ and α be directions through X and A, respectively. Then there is a unique M-transformation which maps X on A, and maps the direction ξ onto the direction α. See Figure 55.1.

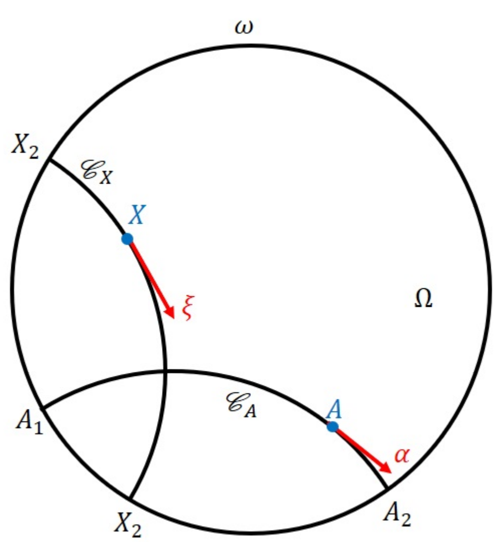

Figure 55.1 (slightly modified)

Corollary 55.A. Let $\mathscr{C}_A$ be a circle through point A of Ω to ω, and let $\mathscr{C}_B$ be a circle through a point B of Ω orthogonal to ω. Then there exist just two M-transformations which map A on B and $\mathscr{C}_A$ on $\mathscr{C}_B$. See Figure 55.2.

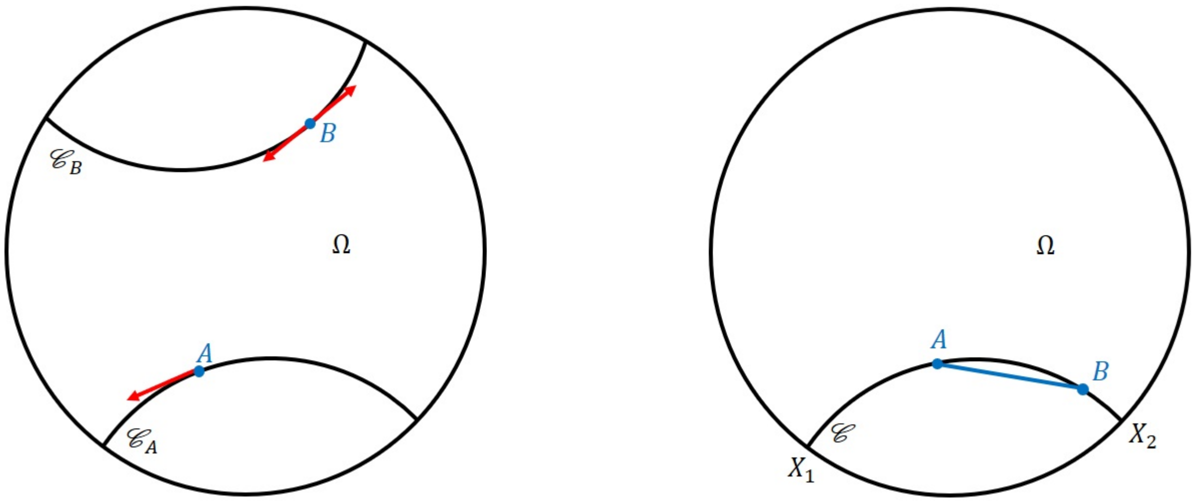

Figure 55.2 (slightly modified)

Note. Corollary 55.A is analogous to Theorem 43.2. Theorem 43.2 involves isometries, though we do not yet have a metric on Ω (other than the Euclidean metric, which is not useful in the new setting of hyperbolic geometry). We will introduce a metric on Ω in Section 58.1. More Hyperbolic Triangles.

Theorem 55.3. There exists a unique M-transformation which interchanges two given points A and B of Ω.

Note. John Playfair (March 10, 1748–July 20, 1819), a professor of natural philosophy at the University of Edinburgh, published a textbook, *Elements of Geometry; containing the first six books of Euclid, with two books on the geometry of solids, to which are added, elements of plane and spherical trigonometry*, in 1860 (this can be read or downloaded from Google Books). In this, Playfair proves that the following:

> **Playfair's Parallel Axiom.** For a given line ℓ and a point P not on ℓ (both in the Euclidean plane), there is exactly one line through point P which is parallel to line ℓ.

This is equivalent to Euclid's Parallel Postulate (the "Fifth Postulate"); see my online notes for Introduction to Modern Geometry (MATH 4157/5157), History of Geometry on Euclid's *Elements*, on Section 2.1. Book I. So if we consider non-Euclidean geometry, then we need to negate Euclid's Parallel Postulate or, equivalently, negate Playfair's Parallel Axiom. We have two options when negating Playfair's Axiom: One in which there are more than one line through P parallel to ℓ, and a second one in which there are no lines through P parallel to ℓ (here, "parallel line" in a plane are those that do not intersect). In hyperbolic geometry, we postulate more than one line through P which is parallel to ℓ. The second option leads to "elliptic geometry." For more details, see my online presentations on A Quick Introduction to Non-Euclidean Geometry, on Hyperbolic Geometry, and on Eudlicean Geometry (this last presentation includes some details on the parallel postulate and its negation using Saccheri quadrilaterals).

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