Section 56. The Poincaré Model

Note. In this section we give several definitions of our model of hyperbolic geometry. Properties will be explored in the next few sections.

Definition. The interior Ω of the unit circle ω in the complex plane is our space of points, called the *p-plane*. A point of Ω is a *point* (or *p-point*). The *lines* or *p-lines* are the arcs of circles in Ω which are orthogonal to ω, or are diameters of ω (that is, line segments in Ω that are orthogonal to ω). If a point in Ω lies on p-line then the point *lies on* the p-line. The collection of p-points and p-lines in Ω for the *Poincaré model* for hyperbolic geometry. See Figure 56.1.

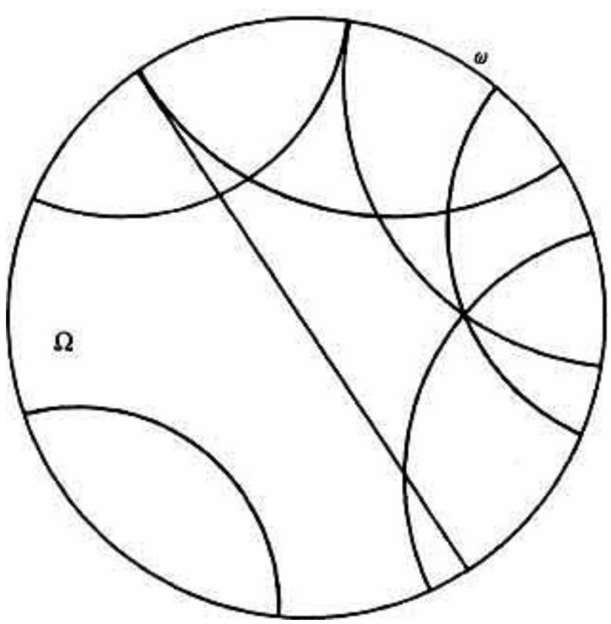

Figure 56.1

Note 56.A. The incidence results of real Euclidean plane geometry hold in this new geometry. We have:

1. Through two distinct p-points there passes a unique p-line.

2. Two distinct p-lines intersect in at most one p-point.

Note. The angle between two p-lines is the usual Euclidean angle as circles or lines in the Euclidean plane. Congruence of angles between p-lines is also dealt with as in the usual Euclidean sense.

Definition. A *p-segment* is the set of p-points on a p-line between two p-points (where the idea of "between" is the same as in the Euclidean sense).

Note. Concepts of congruence are related to an idea of moving one set of points to another place. This is accomplished in the Euclidean setting using isometries. The direct isometries preserve angles and their sense. In the Poincaré Model, the M-transformations play the role of direct isometries in the Euclidean setting. So we use M-transformations in defining congruence of p-segments and p-triangles.

Definition. Two p-segments AB and CD are *p-congruent*, denoted $AB \stackrel{p}{=} CD$, if there is an M-transformation which maps A to C and B to D.

Theorem 56.A. p-congruence of p-segments satisfy:

(i) $AB \stackrel{p}{=} AB$ (reflexive).

(ii) If $AB \stackrel{p}{=} CD$ then $CD \stackrel{p}{=} AB$ (symmetric).

(iii) If $AB \stackrel{p}{=} CD$ and $CD \stackrel{p}{=} EF$ then $AB \stackrel{p}{=} EF$ (transitive).

That is, p-congruence is an equivalence relation. In addition, $AB \stackrel{p}{=} BA$.

Note. Theorem 56.A is to be proved in Exercise 56.3. The proof is based on the fact that the M-transformations form a group (as is shown in Theorem 54.B).

Definition. A *p-triangle* is a triad of points which do not lie on a p-line. Two p-triangles, ABC and $A'B'C'$, are *p-congruent* if there is an M-transformation which maps A to A', B to B', and C to C'.

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