Section 57. Hyperbolic Triangles and Parallels

Note. In this section we consider the Side-Angle-Side Theorem for congruent p-triangles. We consider a new kind of transformation that resembles the M-transformations, but involves a reflection. We show that the Poincaré model represents a geometry different from that of Euclid.

Note. The next result refers to p-triangles with "equal orientation." Since p-triangles are subsets of Ω, then we can use the order of the vertices, A, B, C say, to put an orientation on triangle ABC (clockwise or counterclockwise using the natural "right-hand" orientation of the complex plane, for example). Notice that triangles ABC and ACB have opposite orientations.

Theorem 57.1.I. The SAS Theorem. Given two p-triangles ABC and $A'B'C'$ with equal orientation, the congruences $AB \overset{p}{=} A'B'$, $AC \overset{p}{=} A'C'$, and $\sphericalangle CAB \overset{p}{=} C'A'B'$ imply the p-congruence of the triangles, so that $\sphericalangle ABC \overset{p}{=} \sphericalangle A'B'C'$, $\sphericalangle ACB \overset{p}{=} \sphericalangle A'C'B'$ and $BC \overset{p}{=} B'C'$. See Figure 57.1.

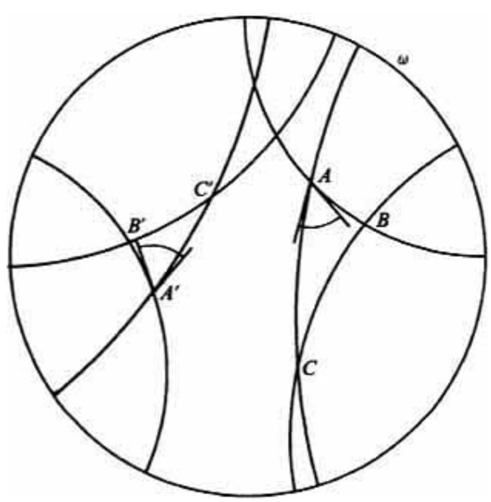

Figure 57.1

Note. Notice that Theorem 57.1.I only addressed triangles with equal orientations. This was not the case in Euclidean geometry (though it would be if we restricted our attention to direct isometries in that setting; see Section 43. The Theorem of Isometries). For example, in Figure 57.2 consider triangles ABC and $A'B'C'$. These triangles are not equally oriented and we cannot slide one triangle around so that it lies exactly on the other (that is, there is not direct isometry, $z' = az + b$ where $|a| = 1$, that transforms one triangle to the other). Now we can slide triangle ABC onto the equally oriented triangle $A^*B^*C^*$. We can also translate triangle $A'B'C'$ onto triangle $A^*B^*C^*$, but we must "turn it over" in the translation. This is accomplished with an indirect isometry of the form $z' = c\overline{z} + d$ where $|c| = 1$. The M-transformations preserve orientation (and the sense of angles), so if we wish to deal with transformations that "turn over" then we need to introduce some type of reflection about a line.

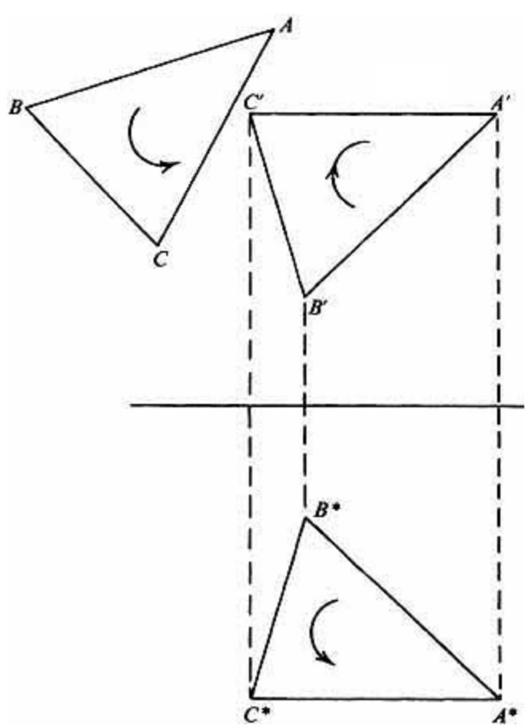

Figure 57.2

Note. In the Euclidean setting, if we restrict ourselves to direct isometries then the SAS theorem cannot be proved for triangles that are not equally oriented. In some axiomatic treatments of Euclidean geometry, this is avoided by taking the Side-Angle-Side theorem as an axiom (so that orientations need not be mentioned). This is the approach taken in my online notes for Introduction to Modern Geometry-Axiomatic Method (MATH 4157/5157); see Section 2.9. The Congruence Postulate and Postulate 16. To deal with reflections in the Poincaré model, we introduce a new type of transformation.

Definition. A transformation of the form $Z = \dfrac{a\overline{z} + b}{c\overline{z} + d}$, where $ad - bc \neq 0$, which maps the unit circle $|z| = 1$ to itself and maps the inside of $|z| = 1$ to itself is a *conjugate M-transformation*. The collection of conjugate M-transformations is denoted *M_--transformations* and the collection of usual M-transformations is denoted (in this setting) *M_+-transformations*. The collection of M_--transformations combined with M_+-transformations is denoted M^*-transformations.

Note. The M_--transformations do not form a group (there is not identity, for example). But the collection of M^*-transformations do form a group (with the M_+-transformations forming a subgroup of this group; this is Exercise 57.6). The M_+-transformations preserve orientations of triangles and the sense of angles, whereas the M_--transformations reverse these. So the composition of two M_--transformations is a M_+-transformation (as is to be shown in Exercise 57.8). We also have for z_1, z_2, z_3, and z_4 complex numbers which are mapped to z_1', z_2', z_3', and z'_4, respectively, under a conjugate Möbius transformation, that the cross-ratio satisfies $(z_1, z_2, z_3, z_4) = \overline{(z'_1, z'_2, z'_3, z'_4)}$ (this is Exercise 57.7).

Definition 57.A. We now **redefine** p-congruence. The p-line segments AB and $A'B'$ are *p-congruent* if there is an M^*-transformation which maps A to A' and maps B to B'. Angles congruent in the Euclidean sense are p-congruent.

Note. With the new definitions of p-congruence of p-line segments and angles, we can give a corresponding new version of the SAS theorem.

Theorem 57.1.II. New SAS Theorem. Given two p-triangles ABC and $A'B'C'$, the congruences $AB \stackrel{p}{=} A'B'$, $AC \stackrel{p}{=} A'C'$ and $\sphericalangle CAB \stackrel{p}{=} \sphericalangle C'A'B'$ imply the p-congruences of the triangles, so that $\sphericalangle ABC \stackrel{p}{=} A'B'C'$, $\sphericalangle ACB \stackrel{p}{=} A'C'B'$ and $BC \stackrel{p}{=} B'C'$.

Note. As observed in Section 55. M-Transformations, Playfair's Parallel Axiom is equivalent to Euclid's Parallel Postulate (the "Fifth Postulate"). That is, we can prove Playfair's result using Euclid's Parallel Postulate, and conversely we can prove Euclid's Parallel Postulate using Playfair's Parallel Axiom (and the other postulates of Euclid). We had:

> **Playfair's Parallel Axiom.** For a given line ℓ and a point P not on ℓ (both in the Euclidean plane), there is exactly one line through point P which is parallel to line ℓ.

Recall that two coplanar lines are *parallel* if they do not intersect. So if we can demonstrate a geometry satisfying the postulates of Euclidean geometry, with the exception of the Parallel Postulate (or equivalently, with the exception of Playfair's Parallel Axiom), then we have an example of a non-Euclidean geometry. The Poincaré model meets these conditions.

Note. In Figure 57.3 p-line m and point A not on m are given in Ω. We see that there is more than one line in Ω which contains point A and is parallel to m (that is, does not intersect m in Ω). In fact, there are infinitely many lines through A that are not parallel to m (think of them rotating about point A until one end or the other intersects m on the boundary ω). Pedoe makes a restriction of the use of the term "p-parallel" to indicate these two extreme lines, as follows below).

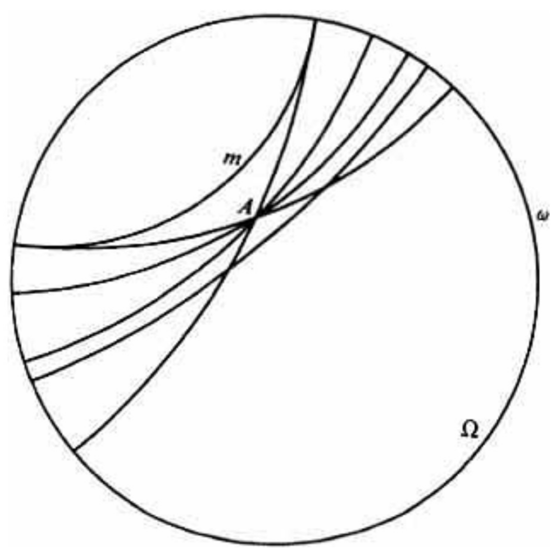

Figure 57.3

Definition 57.B. For a given p-line m and point A in Ω, the two lines described above which pass through point A and intersect m only on ω are *p-parallel* to m.

Note. Definition 57.B implies a difference between the idea of parallel lines not intersecting and the concept of p-parallel. In some settings, Pedoe's "p-parallel" is called "parallel" and nonintersecting lines that are not p-parallel are called "ultraparallel." See my online supplemental notes for Complex Analysis 1 (MATH 5510) on Supplement. Hyperbolic Geometry and the Poincare Disk (notice Definition H.5.2.1). With Pedoe's definition, we have the next result in place of Euclid's Parallel Postulate.

Theorem 57.2. Through any given p-point A which does not lie on a given p-line m there pass exactly two p-lines p-parallel to the given p-line.

Note. Since Theorem 57.2 contradicts Euclid's Parallel Postulate, our p-geometry is a model of non-Euclidean geometry. Pedoe comments (see page 229):

> "Our p-plane is a *model* for such a geometry. If Euclidean geometry is an internally consistent system, without contradictions, then so is the model, and its existence shows that the parallel axiom in Euclidean geometry is *independent* of the other axioms, that is *it cannot be deduced from them*."

The ideas of consistency and independence of axiomatic systems is discussed in my online notes for Introduction to Modern Geometry-Axiomatic Method (MATH 4157/5157) on Section 1.4. Consistency and Section 1.5. Independence.

Revised: 12/20/2021