Section 58. More Hyperbolic Triangles and Distance

Note. In this section we give some properties of triangles in p-geometry, introduce a metric on Ω, and give some properties of the metric and additional properties of triangles using the metric. Now that we know the Parallel Postulate does not hold in p-geometry, then we should not be surprised that other results from Euclidean geometry do not hold (namely, those requiring the Parallel Postulate).

Note. In Euclid's *Elements*, Book I Proposition 32 states: "In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles." In other words, the sum of the angles of a Euclidean triangle is π (in radians). This does not hold in p-geometry, as the next result shows. First, we need a definition.

Definition. A p-line is *convex* to a point $C \in \Omega$ if the p-line is a circle in $\mathbb{C}$ and C is exterior to the circle.

Theorem 58.1. The sum of the angles of a p-triangle ABC is (strictly) less than π.

Note. The next result addresses exterior angles to a triangle. Theorems 58.1 and 58.2 combine to give a result in p-geometry analogous to Euclid's Book I Proposition 32.

Theorem 58.2. In any p-triangle, if one of the p-sides be extended, the exterior angle is greater than either of the interior and opposite angles.

Note. Recall that a metric (or "distance function") on a set S is a mapping of $S \times S$ into $\mathbb{R}$ such that

(1) $d(A, B) = d(B, A) \geq 0$ for all $A, B \in S$,

(2) $d(A, B) = 0$ if and only if $B = A$, and

(3) $d(A, B) + d(B, C) \geq d(A, C)$ for all $A, B, C \in S$ (the Triangle Inequality).

We define a metric on Ω using p-lines and cross-ratios. For the p-line containing points A and B, let α and β be the points of intersection of the p-line with ω, as shown in Figure 58.3.

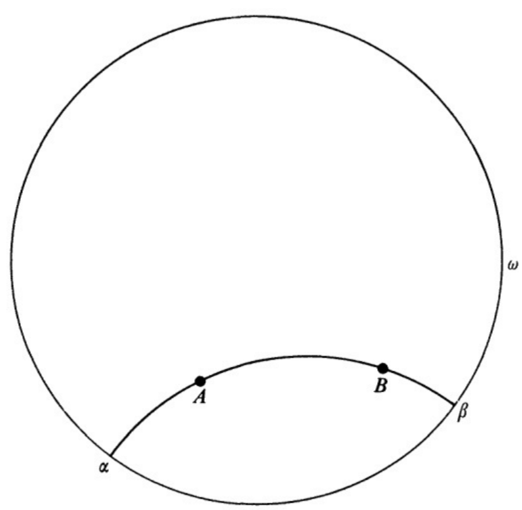

Figure 58.3

First, we need the following preliminary result.

Theorem 58.A. If the points α, A, B, β are in this order on a p-line, where α and β are on ω, then the cross-ratio $(\alpha, \beta; A, B)$ is a positive real number less than 1.

Definition. Let the p-line ab cut ω in the points α and β. Then define

$$d(A, B) = |\log(\alpha, \beta; A, B)|.$$

Note 58.A. Since for distinct points α, β, A, B we have $0 < (\alpha, \beta; A, B) < 1$ by Theorem 58.A, then $\log(\alpha, \beta; A, B) < 0$ and so $|\log(\alpha, \beta; A, B)| > 0$, as needed in part of (1) in the definition of metric. Since $A, B \in \Omega$ and α, β are on ω then $\{A, B\} \cap \{\alpha, \beta\} = \varnothing$. In Exercise 53.6 it is to be shown that interchanging A and B (or interchanging α and β) in $(\alpha, \beta; A, B)$ results in a reciprocal value of the cross-ratio:

$$(\alpha, \beta; B, A) = \frac{1}{(\alpha, \beta; A, B)}.$$

This implies

$$d(B, A) = |\log(\alpha, \beta; B, A)| = |\log(\alpha, \beta; A, B)^{-1}|$$

$$= |-\log(\alpha, \beta; A, B)| = |\log(\alpha, \beta; A, B)| = d(A, B).$$

So the other part of (1) of the definition of metric is satisfied. Now if $A = B$ then (by the definition of cross-ratio)

$$(\alpha, \beta; A, B) = \left(\frac{\alpha - A}{\beta - A}\right) \Big/ \left(\frac{\alpha - B}{\beta - B}\right) = \left(\frac{\alpha - A}{\beta - A}\right) \Big/ \left(\frac{\alpha - A}{\beta - A}\right) = 1,$$

and $d(A, B) = d(A, A) = |\log 1| = 0$, as needed for part of (2) in the definition of metric. If $d(A, B) = |\log(\alpha, \beta; A, B)| = 0$, then $(\alpha, \beta; A, B) = 1$ or

$$\left(\frac{\alpha - A}{\beta - A}\right) \Big/ \left(\frac{\alpha - B}{\beta - B}\right) = 1 \text{ or } \left(\frac{\alpha - A}{\beta - A}\right) = \left(\frac{\alpha - B}{\beta - B}\right)$$

or

$(\alpha - A)(\beta - B) = (\alpha - B)(\beta - A)$ or $\alpha\beta - A\beta - B\alpha + AB = \alpha\beta - B\beta - A\alpha + AB$

or $A(\alpha - \beta) = B(\alpha - \beta)$ or $A = B$, as needed for the other part of (2) in the definition of metric. We establish the Triangle Inequality (3) below (in Theorem 58.2.IV) and then we will have confirmed that d is a metric on Ω. First, we prove a geometrically clear result which is related to the idea that "the shortest path between two points is a line." More precisely, the shortest distance between two points on a manifold with a differential structure (or a "metric form") is attained on a geodesic. See my online notes for Differential Geometry (and Relativity) (MATH 5310) on Section 1.7. Geodesics.

Theorem 58.2.I. If A, B, and C are p-points, in this order, on a p-line then $d(A, C) = d(A, B) + d(B, C)$.

Note. In the next two results we relate the metric d to M^*-transformations.

Theorem 58.B. The distance function d is invariant under M^*-transformations. That is, for $A, B \in \Omega$ and M and M^*-transformation, $d(A, B) = d(M(A), M(B))$.

Theorem 58.C. If for points $A, B \in \Omega$ we have $d(A, B) = d(A', B')$, then there is an M_+-transformation which maps A to A' and maps B to B'.

Note. We now see from Theorem 58.B and 58.C that for any points $A, B, A', B' \in \Omega$, we have $d(A, B) = d(A', B')$ if and only if there is an M^*-transformation mapping A to A' and B to B'. Notice that this is consistent approach to dealing with distances in the Euclidean setting in Chapter V when we used isometries.

Note. We have set up p-geometry in the Poincaré Model as a subset of the Euclidean plane $\mathbb{C}$. We now prove additional results in p-geometry by appealing to the terminology and diagrams from the Euclidean plane (but without use of the Parallel Postulate).

Theorem 58.2.II. In a p-triangle ABC, if $d(A, B) = d(A, C)$, then the p-angles ABC and ACB are congruent.

Note. A proof of Theorem 58.2.II is also to be given in Exercise 58.4 which does not use the General Side-Angle-Side Theorem. We next give a result that is a fundamental to our proof of the Triangle Inequality for d.

Note 58.B. We saw in Note 58.B that for points α, A, B, β as given in Figure 58.3 (above),

$$(\alpha, \beta; A, B) = \left(\frac{\alpha - A}{\beta - A}\right) \Big/ \left(\frac{\alpha - B}{\beta - B}\right).$$

Notice that if we take a limit as $B \to A$ (while maintaining the order of Figure

58.3) then we get:

$$\lim_{B\to A}\left(\frac{\alpha-A}{\beta-A}\right)\Big/\left(\frac{\alpha-B}{\beta-B}\right)=1$$

and

$$\lim_{B\to \beta}\left(\frac{\alpha-A}{\beta-A}\right)\Big/\left(\frac{\alpha-B}{\beta-B}\right)=0.$$

Also,

$$(\alpha,\beta;A,B)=\left(\frac{\alpha-A}{\beta-A}\right)\Big/\left(\frac{\alpha-B}{\beta-B}\right)=\frac{(\alpha-A)(\beta-B)}{(\beta-A)(\alpha-B)}$$

is a *continuous* (real valued) function of B. Now $\left|\dfrac{\beta-B}{\alpha-B}\right|=\dfrac{|\beta-B|}{|\alpha-B|}$ is the ratio of the distance between B and β to the distance between B and α. So if B moves from A to β (along the p-line of Figure 58.3) then $|\beta-B|$ decreases and $|\alpha-B|$ increases so that the ratio $|\beta-B|/|\alpha-B|$ (and hence the cross-ratio) strictly decreases from the limit values of 1 to the 0. Since the negative logarithm function is strictly decreasing, then we have that the distance function $d(A,\cdot)$ (recall that the distance function is the negative logarithm of the cross-ratio) is a *strictly increasing* function and ranges from 0 at $B=A$ to (in the limit) ∞ at $B=\beta$. So on a given p-line containing point A, for any positive real number r there is a point C on the p-line such that $d(A,C)=r$ (this point being to the "β side" of A; of course, we can also argue that there is a second point at distance r from A on the "α side" of A).

Theorem 58.2.III. In any p-triangle ABC, the greatest side is opposite the greatest angle.

Note. As a corollary to Theorem 58.2.III, we see that the right angle of a right-angled p-triangle is opposite the longest side of the p-triangle. We now prove the Triangle Inequality for d, establishing that d actually is a metric.

Theorem 58.2.IV. In any p-triangle ABC, $d(A, B) + d(B, C) > d(A, C)$.

Note. Notice that when we speak of a p-triangle ABC we mean that there is no single p-line containing points A, B, C. In the event that points A, B, C lie of a p-line (in this order), then by Theorem 58.1 we have $d(A, B) + d(B, C) = d(A, C)$. So we can say for any points $A, B, C \in \Omega$ that $d(A, B) + d(B, C) \geq d(A, C)$, as the Triangle Inequality traditionally states.

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