Section 59. Horocycles

Note. In this section we address the similarity of p-triangles, define a p-circle, and p-circles in Ω which are tangent to ω (these are called "horocycles").

Note. We have already seen differences in Euclidean geometry and p-geometry (in the angle sum of a triangle and the status of Playfair's Theorem). The next theorem is another difference. To paraphrase, it says that two p-triangles are similar if and only if they are p-congruent! However, it makes no mention of "similar" and because of this result you are unlikely to encounter similarity in a non-Euclidean geometry.

Theorem 59.1. Two p-triangles are p-congruent if the three angles of the one are respectively equal to the three angles of the other.

Note. Since we have the metric on Ω as defined in Section 58. More Hyperbolic Triangles and Distance, we can define a p-circle in the usual way.

Definition. The locus of a points in Ω which are at a constant p-distance of r from a given p-point A is a *p-circle* with *p-center* A and *p-radius* r.

Note. The next theorem allows us to recognize a p-circle in our model of p-geometry. The proof is partially based on some properties of circles from Chapter

II, and some of the claims are not well-justified.

Theorem 59.2.I. A p-circle, center A is a Euclidean circle orthogonal to the family of p-lines which pass through A.

Note. In Theorem 59.2.I, the p-center of the p-circle (where distances are measure using the p-distance metric d) and the center of the Euclidean circle (where distances are measure in the usual way) do not in general coincide. In fact, they only coincide when the center p-center is O, as is to be shown in Exercise 59.3.

Definition. A Euclidean circle that lies in $\Omega \cup \omega$ and is tangent to ω at a single point is a *horocycle*.

Note. The next result shows that horocycles tangent to ω at the same point behave like concentric Euclidean circles; that is, they cut radial line segments at a constant length.

Theorem 59.2.I. Two horocycles tangent to ω at the same point β cut off equal p-distances on the p-lines through β.

Note. The Poincaré model of p-geometry on Ω is a model for the version of non-Euclidean geometry called *hyperbolic geometry*. In hyperbolic geometry, Playfair's

Theorem is violated (and hence the Euclidean Parallel Postulate is violated) by the existence of more than one parallel to a given line through a given point not on the line. In a different version of non-Euclidean geometry called *elliptic geometry*, Playfair's Theorem is violated in that there are no parallel lines.

Note. A model of elliptic geometry is given in Exercise 59.2:

> **Exercise 59.2.** S is the surface of a sphere in real three-dimensional Euclidean space, and O is the center. ...we have a model of an elliptic non-Euclidean geometry if we take S as our Plane, and identify a *pair* of points on S which are diametrically opposed (that is, [the line joining them] passes through the center) as a Point in the Plane. The Lines in our Plane are to be given by the intersections with S of Euclidean planes through O. [So the Lines are great-circles on S.]

In the next chapter, "The Projective Plane and Projective Space," we'll see other models for elliptic geometry.

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