Chapter VII. The Projective Plane and Projective Space

Note. In this chapter we give a model for "projective geometry" in two and three dimensions. We do so without a mention of distances or angle measure; we primarily concentrate on the idea of incidence. The properties of this geometry are invariant under "projection," thus the name.

Section 60. The Complex Projective Plane

Note. In this section we define "point" and "line" in a model for projective geometry. We show that two points determine a unique line, and any two distinct lines intersect in a unique point. We also describe the Principle of Duality between points and lines in the projective plane

Definition. Consider ordered triples (or "triads") of complex numbers, (y_0, y_1, y_2), where not all of y_0, y_1, y_2 are zero. Define an equivalence relation on the set of all such triples as $(x_0, x_1, x_2) \equiv (y_0, y_1, y_2)$ if and only if there is fixed $k \in \mathbb{C}$ such that $(x_0k, x_1k, x_2k) = (y_0, y_1, y_2)$. A *point* in projective geometry (or in the "complex projective plane") is an equivalence class of ordered triples. We represent the equivalence class containing (y_0, y_1, y_2) simply as "(y_0, y_1, y_2)" (or as one of the other representatives of the equivalence class). The coordinates (y_0, y_1, y_2) of a point in projective geometry are called *homogeneous coordinates*.

Note. The classical triple (y_0, y_1, y_2) represents a point in $\mathbb{C}^3$ distinct from the origin. So we can think of a point in projective geometry as a line in $\mathbb{C}^3$ through the origin (excluding the origin).

Definition. A *line* in the complex projective plane is the subset of points in the plane whose coordinates satisfy a linear homogeneous equation $u^0X_0 + u^1X_1 + u^2X_2 = 0$, where not all the u^i $(i = 0, 1, 2)$ are zero. The points of the line are *incident* with the line.

Note. Notice that if the classical triple (y_0, y_1, y_2) satisfies the linear homogeneous equation, the so does the classical triple (y_0k, y_1k, y_2k) where $k \neq 0$:

$$u^0(y_0) + u^1(y_1) + u^2(y_2) = 0 \Leftrightarrow u^0(y_0k) + u^1(y_1k) + u^2(y_2k) = 0.$$

So a line is well-defined as a collection of points in the complex projective plane.

Note. we now state two theorems. The first shows that lines in the projective plane are uniquely determined by two points, as desired. The second shows that projective geometry is a form of non-Euclidean geometry in which there are no parallel lines. In the supplements, we present the proof of the second theorem first.

Theorem 60.I. In the projective plane, two distinct points determine a unique line with which they are incident.

Theorem 60.II. In the projective plane two distinct lines intersect in a unique point.

Note. We add a third theorem which shows that the geometry is not trivial.

Theorem 60.III. The projective plane contains at least four distinct points, no three of which are collinear.

Definition. Consider the line $u^0X_0 + u^1X_1 + u^2X_2 = 0$. The *coordinates of the line* are $[u^0, u^1, u^2]$ or $[ku^0, ku^1, ku^2]$ for any $k \in \mathbb{C}$, $k \neq 0$.

Note 60.A. Suppose point (y_0, y_1, y_2) lies on line $[u^0, u^1, u^2]$. We say that line $[u^0, u^1, u^2]$ "contains" point (y_0, y_1, y_2), and point (y_0, y_1, y_2) "lies on" line $[u^0, u^1, u^2]$. This produces a *dual* behavior between points and lines. That is, a statement concerning points and lines which uses the relationships "lying on," "passing through," "join," "intersection," "collinear," and "concurrent" can be reworded as a statement concerning lines and points (respectively) by interchanging the terms "lying on" and "passing through," "join" and "intersection," "collinear" and "concurrent." That is, we have the following dualities:

Object/Relationship	**Dual Object/Relationship**
Point	Line
Line	Point
point *Lying On* a line	line *Passing Through* a point
two points *Joined* by a line	two lines *Intersect* at a point
Collinear points	*Concurrent* lines

This is the *Principle of Duality*.

Note. The Desargues Theorem (Theorem 5.2) of plane Euclidean geometry states:

Theorem 5.2. The Desargues Theorem.
If ABC and $A'B'C'$ are triangles with distinct vertices, and AA', BB', and CC' intersect in a point V, then the intersection Q_1, Q_2, and Q_3 of corresponding sides BC and $B'C'$, of corresponding sides CA and $C'A'$, and of corresponding sides AB and $A'B'$ (respectively) are collinear, if these intersections exist. See Figure 5.1.

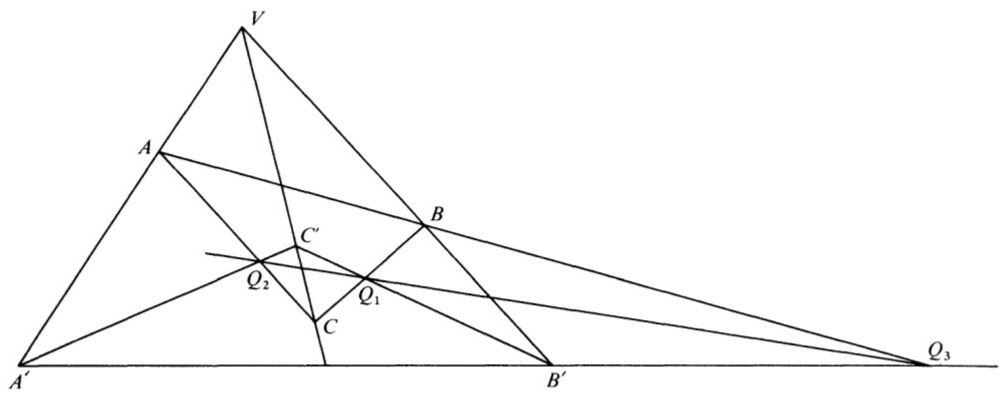

Figure 5.2

We will see the Desargues Theorem in Section 62 in the projective plane setting. Introducing the duality terminology, the Desargues Theorem we replace the statement that "AA', BB', and CC' intersect in a point V," with the statement that "AA', BB', and CC' are *concurrent* (i.e., pass through the same point)." We know the sides intersect by Theorem 60.II, so we can refer to points Q_1, Q_2, and Q_3 as $BC\cap B'C'$, $CA\cap C'A'$, and $AB\cap A'B'$ (respectively), so that the conclusion becomes $BC \cap B'C'$, $CA \cap C'A'$, and $AB \cap A'B'$ are *collinear*. By the Principle of Duality, we therefore have that if abc and $a'b'c'$ are triangles in the projective plane (here a, b, c, a', b', c' represent lines in the projective plane, not points) which are such that points aa', bb', cc' (here, for example, aa' represents the point of intersection of lines a and b) are *collinear* then the lines joining corresponding vertices $bc \cup b'c'$, $ca \cup c'a'$, and $ab \cup a'b'$ (here, for example, $bc \cup b'c'$ represents the line joining points bc and $b'c'$) are *concurrent* (i.e., pass through a point). The choice of notation of A and B for points, and a and b for lines in the dual statement is intentional, as is the replacement of $\cap$ with $\cup$.

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