Section 61. A Model for the Projective Plane

Note. In this section we use the vector space $V_3 = \mathbb{C}^3$ to create a model of projective geometry. We define points and planes in the projective plane model by associating points in V_3 with each other (using an equivalence relation), effectively "mod-ing out" a dimension in V_3. We also discuss a point at infinity and a line at infinity in a real linear space.

Note. In the previous section we used an equivalence relation to associate a line (or "ray") through the origin in $\mathbb{C}$ (minus the origin itself) with a point in the projective plane. With this in mind, we introduce the next definition.

Definition. Let V_3 represent the complex vector space $\mathbb{C}^3 = \{(y_0, y_1, y_2) \mid y_0, y_1, y_2 \in \mathbb{C}\}$. If $y \neq (0,0,0)$ then the *ray* of V_3 is the set of vectors $\{(y_0k, y_1k, y_2k) \mid k \in \mathbb{C}, k \neq 0\}$. We denote this set as $yk = (y_0k, y_1k, y_2k)$. A *point* in the complex projective plane is a ray of V_3.

Note. As in any vector space, two vectors $y = (y_0, y_1, y_2)$ and $z = (z_0, z_1, z_2)$ in V_3 are linearly dependent if there exists complex scalars λ and μ, not both 0, such that $y\lambda + z\mu = 0$ (and are linearly independent otherwise). Notice that, unlike in sophomore Linear Algebra (MATH 2010), we do not notationally distinguish between vectors and scalars. For example, we have just used "0" to denote both the scalar 0 and the vector 0. This is common practice in upper-level classes where the context implies whether we are dealing with vectors or scalars. In Linear Algebra, we would use the notation "0" for scalar zero and "$\vec{0}$" for the zero vector.

Note 60.A. If vectors y and z are linearly independent, then neither is the zero vector and so each determines a ray in V_3. The span of y and z (i.e., the set of all linear combinations of y and z) forms a subspace of V_3. For $X = y\lambda + z\mu$ (that is, for X in the span of x and y), we have the three equations:

$$\begin{aligned} X_0 &= y_0\lambda + z_0\mu \\ X_1 &= y_1\lambda + z_1\mu \\ X_2 &= y_2\lambda + z_2\mu. \end{aligned}$$

As shown in the proof of Theorem 60.1 (with some change in variables), we have $(y_1z_2 - y_2z_1)X_0 + (y_2z_0 - y_0z_2)X_1 + (y_0z_1 - y_1z_0)X_2 = 0$. Notice that this is a linear homogeneous equation in (X_0, X_1, X_2) and can be written as an equation involving a determinant:

$$\begin{vmatrix} X_0 & X_1 & X_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = 0.$$

This equation represents both the plane in V_3 (i.e., the span of two linearly independent vectors in V_3) defined by distinct rays $y\lambda$ and $z\mu$, and the line in the projective plane which joins the two distinct points y and z.

Note. We now have a model for the complex projective plane where *points* are rays in V_3 as defined above (or equivalence classes of points in V_3) and *lines* are as defined in Note 60.A (that is, as planes in V_3).

Definition/Note 60.B. Now consider the real space $\mathbb{R}^3$. Let π be a plane in $\mathbb{R}^3$ and choose a coordinate system in which π does not contain the origin $O = (0, 0, 0)$. A ray through O which is not parallel to plane π intersects π at a unique point P. See Figure 61.1. If the ray is the set of points $\{(y_0k, y_1k, y_2k) \mid k \in \mathbb{R}\}$, then we can define an equivalence relation as above where we associate all points in the set with each other.

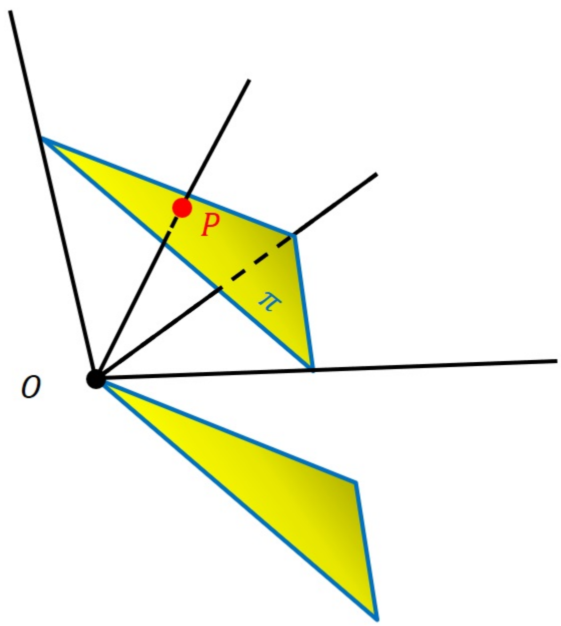

Figure 61.1 (revised)

This differs from the approach above, in that rays through O may be parallel to π. If ray $(z_0, z_1, z_2)k$ is parallel to π the we say that the point $(z_0, z_1, z_2)$is a *point at infinity* (or an *ideal point in* π). If plane π has equation $u^0X_0 + u^1X_1 + u^2X_2 = C$ where C is a constant, then the equation of the plane through O which is parallel to π is $u^0X_0+u^1X_1+u^2X_2 = 0$ (recall the formula for a plane in $\mathbb{R}^3$ and the orthogonal vector to such a plane; see my online Calculus 3 [MATH 2110] on Section 12.5. Lines and Planes in Space). Notice that this is also the formula for a line in the projective space. We call this line the *line at infinite in the plane* π.

Note. We now return to the complex projective space. This spaced is based on vectors in V_3 (well, technically on equivalence classes of vectors in V_3). Of course $V_3 = \mathbb{C}^3$ is a 3-dimensional space and has as its standard basis vectors $E_0 = (1, 0, 0)$, $E_1 = (0, 1, 0)$, and $E_3 = (0, 0, 1)$. Recall that, in general, a basis of a vector space is a linearly independent spanning space (see Definition 3.6 in my online notes for Linear Algebra [MATH 2010] on Section 3.2. Basic Concepts of Vector Spaces). Since V_3 is dimension three, then any linearly independent set of three vectors forms a basis for V_3. Therefore, if $E_)^*$, E_1^*, and E_2^* are linearly independent vectors in V_3, the any vector $y \in V_3$ can be uniquely written as a linear combination of the form $y = E_0^*\lambda + E_1^*\mu + E_2^*\nu$ (see Theorem 3.3, Unique Combination Criterion for a Basis, in the Linear Algebra notes just mentioned). We continue to follow Pedoe's presentation, but a warning is appropriate. In a three dimensional vector space, we represent *vectors* as an ordered triple of scalars (this is the "coordinate vector" of the vector with respect to an ordered basis; see Section 3.3. Coordinatization of Vectors in my online Linear Algebra notes). However, we also represent *points* in a 3-dimensional space with an ordered triple. Pedoe is blurring the lines between points and vectors! The notation is the same, but **points and vectors** in a finite dimensional space **are not the same**! You can add vectors together, but you cannot add points together. Vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ have magnitude and direction, but points in $\mathbb{R}^n$ and $\mathbb{C}^n$ do not. Points in $\mathbb{R}^n$ and $\mathbb{C}^n$ have a location (in a sense, they are nothing but location), but vectors in $\mathbb{R}^n$ and $\mathbb{C}^n$ do not have a location. The proper way to address $\mathbb{R}^n$ and $\mathbb{C}^n$ are as vector spaces. We can then put a *geometric interpretation* on these vector spaces and then we can "draw pictures." It's the geometric interpretation that allows us to discuss lines, rays, and planes.

Note. With π as a plane in V_3, if we choose three noncollinear points in π then these three points determine three vectors in V_3 (these vectors are in "standard position" with their tail at O and head at the point in π). Every point P in π can be written as $P = E_0^*\lambda + E_1\mu + E_2^*\nu$. Think of P not as a point, but as a vector. We can use "vectors" E_0^*, E_1^*, and E_2^* (with the components of the vectors the same as the coordinates of the corresponding points) to express plane π as the translation of a vector space. Two linearly independent vectors parallel to π are $E_0^* - E_1^*$ and $E_0^* - E_2^*$ (think geometrically here), and a translation vector is any one of E_0^*, E_1^*, or E_2^*. So any point P (interpreted as a vector) is of the form $P = E_0^* + (E_0^* - E_1^*)k_1 + (E_0^* - E_2^*)k_2$ (see Definition 2.6 of k-flat in my online Linear Algebra notes on Section 2.5. Lines, Planes, and Other Flats). That is, $P = E_0^*\lambda + E_1^*\mu + E_2^*\nu$ for some $\lambda, \mu, \nu \in \mathbb{C}$, as claimed.

Definition. Let π be a plane in V_3. If E_0^*, E_1^*, and E_2^* are noncollinear points in π, then the triangle formed by E_0^*, E_1^*, E_2^* is the *triangle of reference*. For point P in π, where $P = E_0^*\lambda + E_1^*\mu + E_2^*\nu$, the triple (λ, μ, ν) is the *coordinated of P with respect to the given triangle of reference.*

Note. Recall that "point P" in the projective plane is actually an equivalence class of points in V_3 and so every triple of the form $(\lambda k, \mu k, \nu k)$, $k \neq 0$, is also a coordinate of P; in fact, "the" coordinates of P form an equivalence class (also, there was no uniqueness claim in the choice of λ, μ, and ν.

Example. Let ABC be three noncollinear points in V_3 and let P be a point in the plane π determined by A, B, C which is inside the triangle ABC. The lines AP, BP, and CP cut the opposites sides of triangle ABC in points L, M, and N respectively. See Figure 4.3. Since point L is on line BC then $L = xB + x'C$ for some x and x'. Similarly, $M = yC + y'A$ for some y, y', and $N = zA + z'B$ for some z, z'.

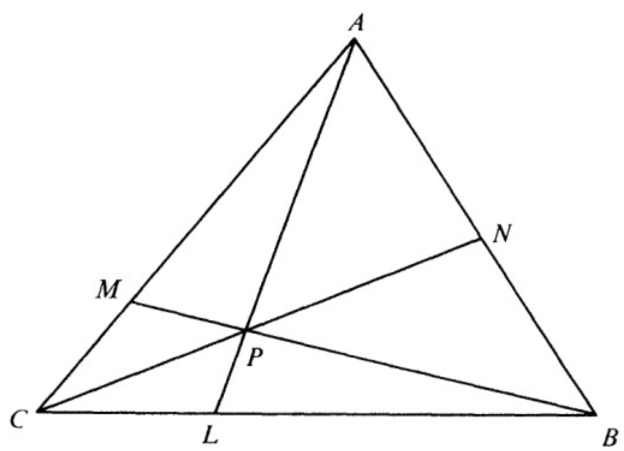

Figure 4.3

If $P = (p, q, r)$ with respect to the triangle of reference ABC, then $L = (0, q, r)$, $M = (p, 0, r)$, and $N = (p, q, r)$. Also, $L = (0, x, x')$, $M = (y', 0, y)$, and $N = (z, z', 0)$. So we ahve the ratios:

$$x : X' = q : r, \; y : y' = r : p, \; z : z' = p : q,$$

and therefore $(x : x')(y : y')(z : z') = (q : r)(r : p)(p : q) = 1$. Hence we have $xyz = x'y'z'$. This is the Theorem of Ceva (Theorem 4.3) in the projective plane setting.